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ON (ω)LINDEL¨OF AND (ω)METACOMPACT TOPOLOGICAL SPACES

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ABSTRACT

We introduce and study several new notions in the framework of (ω)topological spaces – sets equipped with a countably increasing sequence of topologies. Specifically, we introduce (ω)Lindel¨of spaces (every (ω)open cover has a countable subcover), strongly (ω)Lindel¨of spaces, (ω)σ-compact spaces, and (ω)metacompact spaces (every (ω)open cover has, for some n, a point-finite (J_n)open refinement). We also introduce the notion of (ω)continuity and use it to show that (ω)continuous surjective images of (ω)Lindel¨of spaces are (ω)Lindel¨of. A central result (Theorem 3.12) establishes that every strongly (ω)Lindel¨of space that is (J_n)regular for each $n \in N$ is (ω)paracompact, the (ω)topological analogue of the classical theorem that a regular Lindel¨of space is paracompact. We further prove that the product of an (ω)Lindel¨of space with an (ω)compact space is (ω)Lindel¨of, and that a space which is simultaneously (ω)Lindel¨of, (ω)metacompact and (J_n)normal for each $n \in N$ is countably (ω)paracompact. Hereditary properties are also examined.

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KEY WORDS*: (ω)topological space, (ω)Lindel¨of, strongly (ω)Lindel¨of, (ω)continuous, (ω)σ-compact, (ω)metacompact, countably (ω)metacompact.*

1 INTRODUCTION

The notion of an (ω)topological space was introduced and systematically studied by Bose and Tiwari in [1]. Recall that an (ω)*topological space* is a pair $(X, \{J_n\})$ where X is a non-empty set and $\{J_n\}_{n\in N}$ is an increasing sequence of topologies on X satisfying $J_n \subset J_{n+1}$ for every $n \in N$. Such spaces arise naturally in digital topology and in the modelling of infinite neural networks, as elucidated in [2].

Following the foundational work of [1, 2], a wide programme of research on (ω)topological spaces has been carried out. The notions of (ω)compactness and (ω)paracompactness and their variants have been thoroughly investigated in [1, 2, 3, 5, 6]. Countable (ω)paracompactness was introduced and studied in [5, 6]. Near (ω)compactness and almost (ω)compactness appear in [7, 8], while almost (ω)regularity and various separation axioms have been examined in [10, 11, 12]. Connectedness and hyperconnectedness in the (ω) setting are studied in [4, 9].

The present paper contributes several new notions and results to this programme. We introduce and investigate the following new concepts.

(i) (ω)**Lindel¨of spaces** (Definition 3.1): every (ω)open cover of X has a countable subcover.
(ii) **Strongly** (ω)**Lindel¨of spaces** (Definition 3.2): every (ω)open cover of X has, for some $n \in N$, a countable (J_n)open refinement.
(iii) (ω)σ**-compact spaces** (Definition 3.3): X is a countable union of (ω)compact subsets.
(iv) (ω)**continuous maps** (Definition 3.4): the preimage of every (ω)open set is (ω)open.
(v) (ω)**metacompact spaces** (Definition 4.1): every (ω)open cover has, for some $n \in N$, a (J_n)open point-finite refinement.
(vi) **Countably** (ω)**metacompact spaces** (Definition 4.2).

Our principal results establish the following chains of implications (each inclusion being strict, as shown by examples):

$$(\omega)\text{compact} \Rightarrow (\omega^s)\text{Lindelo¨f} \Rightarrow (\omega)\text{Lindel¨of},$$

$$(\omega)\text{compact} \Rightarrow (\omega)\text{paracompact} \Rightarrow (\omega)\text{metacompact} \Rightarrow \text{countably } (\omega)\text{metacompact}.$$

The key theorem of Section 3 (Theorem 3.8) states that every strongly (ω)Lindel¨of space that is (J_n)regular for each $n \in N$ is (ω)paracompact, the (ω)analogue of the classical result that a regular Lindelo¨f space is paracompact [18, 16]. A product theorem for (ω)Lindelo¨f spaces (Theorem 3.7) is also established. The main theorem of Section 4 (Theorem 4.8) shows that a space which is both (ω)Lindelo¨f and (ω)metacompact and satisfies (J_n)normality for each n is countably (ω)paracompact.

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The paper is organised as follows. Section 2 recalls necessary preliminary definitions and known results. Section 3 develops the theory of (ω)Lindelo¨f spaces, introduces (ω)continuity, and establishes the key paracompactness theorem. Section 4 introduces and studies (ω)metacompact spaces. Section 5 summarises the relationships between all notions.

2 PRELIMINARIES

Throughout this paper N denotes the set of positive integers and R the set of real numbers. For an (ω)topological space $(X,\{\mathcal{J}_n\})$ we often write simply X when the sequence of topologies is understood.

We recall the definitions and results from the existing literature that are needed in the sequel.

Definition 2.1 ([1]). If $\{\mathcal{J}_n\}$ is an increasing sequence of topologies on a non-empty set X—that is, $\mathcal{J}_n \subset \mathcal{J}_{n+1}$ for all $n \in$ N—then the pair $(X,\{\mathcal{J}_n\})$ is called an (ω)*topological space*.

Definition 2.2 ([1]). Let $(X,\{\mathcal{J}_n\})$ be an (ω)topological space.

(a) A set $G \subseteq X$ is (ω)*open* if $G \in \mathcal{J}_n$ for some $n \in$ N.

(b) A set $F \subseteq X$ is (ω)*closed* if its complement $X \setminus F$ is (ω)open.

(c) The (ω)*closure* of $A \subseteq X$, denoted $(\omega\text{-cl})A$, is the intersection of all (ω)closed sets containing A.

Remark 2.1. Every finite intersection of (ω)open sets is (ω)open. Indeed, if $G_i \in \mathcal{J}_{n_i}$ for $i = 1,...,k$, set $m = \max_i n_i$; then $G_i \in \mathcal{J}_{n_i} \subset \mathcal{J}_m$ for each i, so $\bigcap_{i=1}^{k} G_i \in \mathcal{J}_m$.

Definition 2.3 ([1]). An (ω)topological space $(X,\{\mathcal{J}_n\})$ is said to be:

(a) (ω)*compact* if every (ω)open cover of X has a finite subcover;

(b) (ω)*Hausdorff* if for any two distinct points $x,y \in X$ there exists $n \in$ N and $U,V \in \mathcal{J}_n$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$;

(c) $(\mathcal{J}_n)$*regular* if the classical topological space $(X,\mathcal{J}_n)$ is regular;

(d) $(\mathcal{J}_n)$*normal* if the classical topological space $(X,\mathcal{J}_n)$ is normal.

Definition 2.4 ([2]). An (ω)topological space $(X,\{\mathcal{J}_n\})$ is said to be (ω)*paracompact* if every (ω)open cover of X has, for some $n \in$ N, a $(\mathcal{J}_n)$open locally finite refinement.

Definition 2.5 ([5, 6]). An (ω)topological space $(X,\{\mathcal{J}_n\})$ is said to be *countably* (ω)*paracompact* if every countable (ω)open cover of X has, for some $n \in$ N, a $(\mathcal{J}_n)$open locally finite refinement.

Definition 2.6 ([6]). Let $(X,\{\mathcal{A}_n\})$ and $(Y,\{\mathcal{B}_n\})$ be two (ω)topological spaces. For each $n \in$ N let $\mathcal{P}_n$ denote the product topology of $\mathcal{A}_n$ and $\mathcal{B}_n$ on $X\times Y$. The pair $(X\times Y,\{\mathcal{P}_n\})$ is called the *product* (ω)*topological space* of $(X,\{\mathcal{A}_n\})$ and $(Y,\{\mathcal{B}_n\})$.

We state the following known theorems for use in the sequel.

Theorem 2.1 ([5, Theorem 2.6]). *Let* $(X,\{\mathcal{J}_n\})$ *be an* (ω)*topological space in which* $(X,\mathcal{J}_n)$ *is normal for each* $n \in$ N. *Then the following statements are equivalent.*

(a) *X is countably (ω)paracompact.*

(b) *Every countable (ω)open cover has, for some $n \in$ N, a point-finite $(\mathcal{J}_n)$open refinement.*

(c) *Every countable (ω)open cover $\{U_i \mid i \in$ N$\}$ has, for some $n \in$ N, a $(\mathcal{J}_n)$open refinement $\{V_i \mid i \in$ N$\}$ satisfying $(\mathcal{J}_n)\mathrm{cl}(V_i) \subseteq U_i$ for each $i \in$ N.*

Theorem 2.2 ([6, Theorem 2.7]). *A topological space $(X,\mathcal{J})$ is normal if and only if for every point-finite $(\mathcal{J})$open cover $\{U_\alpha \mid \alpha \in A\}$ of X there exists a $(\mathcal{J})$open cover $\{V_\alpha \mid \alpha \in A\}$ of X satisfying $(\mathcal{J})\mathrm{cl}(V_\alpha) \subseteq U_\alpha$ for each $\alpha \in A$.*

Theorem 2.3 ([6, Theorem 3.4]). *If $(X,\{\mathcal{A}_n\})$ is (ω)paracompact, $(Y,\{\mathcal{B}_n\})$ is (ω)compact, and both spaces satisfy condition (α) of [6], then the product $(X\times Y,\{\mathcal{P}_n\})$ is (ω)paracompact.*

Theorem 2.4 (Classical; see [18, 16]). *Every regular Lindel¨of topological space is paracompact.*

3 (ω)LINDEL¨OF SPACES

3.1 New definitions

Definition 3.1. An (ω)topological space $(X,\{\mathcal{J}_n\})$ is said to be (ω)*Lindel¨of* if every (ω)open cover of X has a countable subcover.

Definition 3.2. An (ω)topological space $(X,\{\mathcal{J}_n\})$ is said to be *strongly* (ω)*Lindel¨of* (or (ω^s)*Lindel¨of*) if for every (ω)open cover $\mathcal{U}$ of X there exists $m \in$ N such that $\mathcal{U}$ has a countable $(\mathcal{J}_m)$open refinement.

Definition 3.3. An (ω)topological space $(X,\{\mathcal{J}_n\})$ is said to be (ω)*σ-compact* if $X = \bigcup_{i=1}^{\infty} K_i$ for some sequence of (ω)compact sets $K_1,K_2,...$

Definition 3.4. A map $f : (X,\{\mathcal{J}_n\}) \to (Y,\{\mathcal{K}_n\})$ between (ω)topological spaces is called (ω)*continuous* if the preimage $f^{-1}(V)$ is (ω)open in X for every (ω)open set $V \subseteq Y$.

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Remark 3.1. Definition 3.4 is the natural generalisation of classical continuity to the (ω) setting: a map is (ω)continuous if and only if it is continuous with respect to the (ω)open sets on both domain and codomain. The notion is new; it has not appeared in the previous literature on (ω)topological spaces.

3.2 First examples and the implication chain

Theorem 3.1. *Every (ω)compact space is strongly (ω)Lindel¨of, and every strongly (ω)Lindel¨of space is (ω)Lindel¨of.*

Proof. *(ω)compact ⇒ strongly (ω)Lindel¨of.* Let U be any (ω)open cover of the (ω)compact space X. By compactness there is a finite subcover $\{U_1,...,U_k\}$ with $U_i \in J_{n_i}$. Set $m = \max_{1\le i\le k} n_i$; then $U_i \in J_m$ for every i, and $\{U_1,...,U_k\}$ is a finite (hence countable) (J_m)open refinement of U.

Strongly (ω)Lindel¨of ⇒ (ω)Lindel¨of. Let U be any (ω)open cover and let $\{V_i \mid i \in N\}$ be a countable (J_m)open refinement of U for some m. For each i choose $U_i \in U$ with $V_i \subseteq U_i$. Then $\{U_i \mid i \in N\}$ is a countable subcover of U. □

The following example shows that both implications in Theorem 3.1 are strict.

Example 3.1. Let $X = N$ and for each $n \in N$ define

$$J_n = \{\emptyset, N\} \cup P(\{1,2,...,n\}),$$

where $P(A)$ denotes the power set of A. Since $J_n \subset J_{n+1}$, the pair $(N,\{J_n\})$ is an (ω)topological space. We establish:

(a) X is (ω)Lindelo¨f.
(b) X is *not* (ω)compact.
(c) X is *not* strongly (ω)Lindelo¨f.

Proof of (a). Given any (ω)open cover U, for each $k \in N$ select $U_k \in U$ with $k \in U_k$. Then $\{U_k \mid k \in N\}$ is a countable subcover.

Proof of (b). The cover $U = \{\{k\} \mid k \in N\}$ is (ω)open (since $\{k\} \in J_k$) but admits no finite subcover.

Proof of (c). For the same cover $U = \{\{k\} \mid k \in N\}$ and any fixed $m \in N$, every (J_m)open set that is contained in some $\{k\}$ must itself be $\{k\}$ for some $k \le m$. Consequently no collection of (J_m)open sets can cover $N\backslash\{1,...,m\}$ while refining U, and so U admits no countable (J_m)open refinement for any m.

Example 3.2. The space $(N,\{J_n\})$ of Example 3.1 is $(\omega)\sigma$-compact: since $\mathbb{N} = \bigcup_{k=1}^{\infty}\{k\}$ and each singleton $\{k\}$ is (trivially) (ω)compact, the space is a countable union of (ω)compact sets. Nevertheless it is not (ω)compact, as shown in Example 3.1(b).

3.3 Fundamental properties of (ω)Lindel¨of spaces

Proposition 3.2. *If $(X,\{J_n\})$ is (ω)Lindel¨of then, for each $n \in N$, the classical topological space (X,J_n) is Lindel¨of.*

Proof. Fix $n \in N$ and let $\{G_\alpha \mid \alpha \in A\}$ be a (J_n)open cover of X. Since $J_n \subset {}^{S}{}_{k\in N} J_k$, each G_α is (ω)open, so $\{G_\alpha\}$ is an (ω)open cover of X. The (ω)Lindel¨of property yields a countable subcover. □

Theorem 3.3. *Every (ω)closed subspace of an (ω)Lindel¨of space is (ω)Lindel¨of.*

Proof. Let Y be (ω)closed in the (ω)Lindel¨of space X. Let $\{V_\alpha \mid \alpha \in A\}$ be an (ω)open cover of Y, where each $V_\alpha = U_\alpha \cap Y$ for some (ω)open set $U_\alpha \subseteq X$. The family $\{U_\alpha \mid \alpha \in A\} \cup \{X \setminus Y\}$ is an (ω)open cover of X (since $X \setminus Y$ is (ω)open by hypothesis). By the (ω)Lindelo¨f property of X, this cover has a countable subcover. Retaining only those U_α that appear in the subcover and intersecting with Y yields a countable subcover of $\{V_\alpha\}$. □

Theorem 3.4. *A countable union of (ω)Lindel¨of subspaces is (ω)Lindel¨of.*

Proof. Suppose $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is an (ω)Lindelo¨f subspace. Given any (ω)open cover U of X, the family $\{U \cap X_i \mid U \in U\}$ is an (ω)open cover of X_i for each i. By the (ω)Lindel¨of property of X_i there exists a countable $U_i \subseteq U$ such that $\{U \cap X_i \mid U \in U_i\}$ covers X_i. Then $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ is a countable subcover of U. □

Theorem 3.5. *Every $(\omega)\sigma$-compact space is (ω)Lindel¨of.*

Proof. Write $X = \bigcup_{i=1}^{\infty} K_i$ where each K_i is (ω)compact. By Theorem 3.1, every (ω)compact space is (ω)Lindelo¨f, so each K_i is (ω)Lindel¨of. The conclusion follows from Theorem 3.4. □

Theorem 3.6. *Let $f : (X,\{J_n\}) \to (Y,\{K_n\})$ be an (ω)continuous surjection. If X is (ω)Lindel¨of then Y is (ω)Lindel¨of.*

Proof. Let $\{V_\alpha \mid \alpha \in A\}$ be any (ω)open cover of Y. Since f is (ω)continuous, $\{f^{-1}(V_\alpha) \mid \alpha \in A\}$ is an (ω)open cover of X. The (ω)Lindel¨of property of X yields a countable subcover $\{f^{-1}(V_{\alpha i}) \mid i \in N\}$. Since f is surjective, $\{V_{\alpha i} \mid i \in N\}$ covers Y.
□

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3.4 Product theorem

Theorem 3.7. *Let* $(X,\{A_n\})$ *be* (ω)*Lindel¨of and* $(Y,\{B_n\})$ *be* (ω)*compact. Then the product* (ω)*topological space* $(X\times Y,\{P_n\})$ *is* (ω)*Lindel¨of.*

Proof. Let U be an (ω)open cover of $X\times Y$. Replacing members by basic open sets we may assume every $U\in$ U has the form $D\times H$ with $D\in A_{kU}$ and $H\in B_{kU}$ for some $k_U\in N$.

Step 1: Compact slices. For each $x\in X$, the slice $\{x\}\times Y$ is homeomorphic to Y in the product (ω)topology and is therefore (ω)compact. Hence there is a finite subcollection $\mathcal{F}_x=\left\{D_1^x\times H_1^x,\ldots,D_{r(x)}^x\times H_{r(x)}^x\right\}\subseteq\mathcal{U}$ covering $\{x\}\times Y$. Set $m_x=\max_{1\le i\le r(x)}k_{D}ix\times_{H}ix$. Since $A_{ki}\subseteq A_{mx}$ for every i, the finite intersection

$$W_x=D_1^x\cap D_2^x\cap\cdots\cap D_{r(x)}^x$$

belongs to A_{mx} and therefore is (ω)open in X, with $x\in W_x$.

Step 2: Slabs are covered. For any $(z,y)\in W_x\times Y$ there exists i such that $(x,y)\in D_i^x\times H_i^x$, giving $y\in H_i^x$. Since $z\in W_x\subseteq D_i^x$, we obtain $(z,y)\in D_i^x\times H_i^x\in F_x$.

Step 3: Apply (ω)*Lindel¨of.* The family $\{W_x\mid x\in X\}$ is an (ω)open cover of X. Since X is (ω)Lindel¨of there is a countable subcover $\{W_{xj}\mid j\in N\}$. Then $^{S}{}_{j\in N}F_{xj}$ is a countable subfamily of U covering $X\times Y$. □

Remark 3.2. Unlike the product theorem for (ω)paracompact spaces (Theorem 2.3), Theorem 3.7 requires no additional condition such as condition (α). This is because extracting a countable subcover places no constraint on the topology levels of the covering sets.

3.5 From strongly (ω)Lindel¨of to (ω)paracompact

Theorem 3.8. *If* $(X,\{J_n\})$ *is strongly* (ω)*Lindel¨of and* (J_n)*regular for each* $n\in N$*, then* X *is* (ω)*paracompact.*

Proof. Let U be any (ω)open cover of X. By strong (ω)Lindelo¨f property there exists $m\in N$ and a countable (J_m)open refinement $V=\{V_i\mid i\in N\}$ of U. We show that (X,J_m) is a regular Lindel¨of topological space.

- *Regularity:* (X,J_m) is regular by the hypothesis (J_m)regular.
- *Lindel¨of property:* Since X is strongly (ω)Lindelo¨f it is (ω)Lindelo¨f (Theorem 3.1), and Proposition 3.2 then gives that (X,J_m) is Lindelo¨f.

By the classical theorem (Theorem 2.4), the regular Lindel¨of space (X,J_m) is paracompact. Hence the (J_m)open cover V of X admits a (J_m)locally finite (J_m)open refinement W. Since W refines V and V refines U, the collection W is a (J_m)locally finite (J_m)open refinement of U. Hence X is (ω)paracompact. □

Corollary 3.9. *If* $(X,\{J_n\})$ *is strongly* (ω)*Lindel¨of and* (J_n)*normal for each* $n\in N$*, then* X *is* (ω)*paracompact.*

Proof. Every normal space is regular, so this follows directly from Theorem 3.8.

□

4 (ω)METACOMPACT SPACES

4.1 Definitions and basic properties

Definition 4.1. An (ω)topological space $(X,\{J_n\})$ is said to be (ω)*metacompact* if every (ω)open cover of X has, for some $n\in N$, a (J_n)open *point-finite* refinement; that is, a refinement in which every point of X belongs to at most finitely many members.

Definition 4.2. An (ω)topological space $(X,\{J_n\})$ is said to be *countably* (ω)*metacompact* if every countable (ω)open cover of X has, for some $n\in N$, a (J_n)open point-finite refinement.

Theorem 4.1. *The following implications hold:*

$$(\omega)\text{paracompact}\Rightarrow(\omega)\text{metacompact}\Rightarrow\text{countably }(\omega)\text{metacompact.}$$

Proof. Every locally finite collection is point-finite, giving the first implication. The second is immediate since every (ω)open cover contains every countable (ω)open cover as a special case. □

Theorem 4.2. *Every* (ω)*compact space is* (ω)*metacompact.*

Proof. Let U be an (ω)open cover of the (ω)compact space X. By compactness there is a finite subcover $\{U_1,\ldots,U_k\}$ with $U_i\in J_{ni}$. Setting $m=\max_i n_i$, we have $\{U_1,\ldots,U_k\}\subset J_m$, and every finite collection is point-finite. □

The next example shows that (ω)Lindelo¨f does not imply (ω)metacompact, so the two hierarchies are independent.

Example 4.1. The space $(N,\{J_n\})$ of Example 3.1 is (ω)Lindelo¨f (Example 3.1(a)) but *not* (ω)metacompact. Indeed, for the (ω)open cover $U=\{\{k\}\mid k\in N\}$ and any fixed $m\in N$, every (J_m)open set contained in a member of U must be one of $\{1\},\{2\},\ldots,\{m\}$. No such collection can cover $N\backslash\{1,\ldots,m\}$, so U has no (J_m)open point-finite refinement covering N for any $m\in N$.

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4.2 Hereditary properties

Theorem 4.3. *Every* (ω)*closed subspace of an* (ω)*metacompact space is* (ω)*metacompact.*

Proof. Let Y be (ω)closed in the (ω)metacompact space X, and let $\{V_\alpha \cap Y \mid \alpha \in A\}$ be an (ω)open cover of Y where each V_α is (ω)open in X. The family $\{V_\alpha \mid \alpha \in A\} \cup \{X \setminus Y\}$ is an (ω)open cover of X. By (ω)metacompactness of X there exist $n \in \mathbb{N}$ and a ($\mathcal{J}_n$)open point-finite refinement $\{W_\beta \mid \beta \in B\}$. We claim that $\mathrm{R} := \{W_\beta \cap Y \mid \beta \in B,\ W_\beta \cap Y \neq \emptyset\}$ is a ($\mathcal{J}_n|_Y$)open point-finite refinement of $\{V_\alpha \cap Y\}$.

- *Openness.* Each $W_\beta \cap Y$ is open in the subspace topology $\mathcal{J}_n|_Y = \{G \cap Y \mid G \in \mathcal{J}_n\}$.
- *Refinement.* If $W_\beta \not\subseteq X \setminus Y$, then $W_\beta \subseteq V_\alpha$ for some α, so $W_\beta \cap Y \subseteq V_\alpha \cap Y$.
- *Point-finiteness.* For any $y \in Y$, y belongs to at most finitely many W_β (pointfiniteness in X), hence to at most finitely many members of R.

Thus Y is (ω)metacompact. □

Lemma 4.4. *In an* (ω)*Hausdorff space, every* (ω)*compact subset is* (ω)*closed.*

Proof. Let K be (ω)compact in the (ω)Hausdorff space X, and let $x \in X \setminus K$. For each $y \in K$ there exist $n_y \in \mathbb{N}$ and $U_y, V_y \in \mathcal{J}_{n_y}$ with $y \in U_y$, $x \in V_y$, $U_y \cap V_y = \emptyset$. The family $\{U_y \mid y \in K\}$ is an (ω)open cover of K. By (ω)compactness of K there is a finite subcover $\{U_{y1}, \dots, U_{yr}\}$. Set $m = \max_{1 \le i \le r} n_{yi}$. The set $V = \bigcap_{i=1}^{r} V_{y_i}$ belongs to $\mathcal{J}_m$, satisfies $x \in V$, and $V \cap K \subseteq V \cap \left(\bigcup_i U_{y_i}\right) = \emptyset$ (since $V \subseteq V_{yi}$ and $U_{yi} \cap V_{yi} = \emptyset$ for each i). Hence $X \setminus K$ is (ω)open and K is (ω)closed. □

Theorem 4.5. *In an* (ω)*Hausdorff* (ω)*metacompact space, every* (ω)*compact subset is* (ω)*metacompact.*

Proof. By Lemma 4.4, every (ω)compact subset K is (ω)closed. By Theorem 4.3, K is (ω)metacompact. □

4.3 Normality of (ω)compact (ω)Hausdorff spaces

Theorem 4.6. *Every* (ω)*compact* (ω)*Hausdorff space is* (ω)*normal.*

Proof. Let A and B be disjoint (ω)closed subsets of the (ω)compact (ω)Hausdorff space X. Since X is (ω)compact, every (ω)closed subset of X is (ω)compact; hence A and B are both (ω)compact.

Step 1. Fix $a \in A$. For each $b \in B$, the (ω)Hausdorff condition gives $n_{a,b} \in \mathbb{N}$ and $U_{a,b}, V_{a,b} \in \mathcal{J}_{n_{a,b}}$ with $a \in U_{a,b}$, $b \in V_{a,b}$, $U_{a,b} \cap V_{a,b} = \emptyset$. The family $\{V_{a,b} \mid b \in B\}$ is an (ω)open cover of B. By (ω)compactness of B there is a finite subcover $\{V_{a,b1}, \dots, V_{a,bs}\}$. Let $m_a = \max_i n_{a,bi}$ and set

$$W_a = \bigcap_{i=1}^{s} U_{a,b_i} \in \mathcal{J}_{m_a}, \qquad V^a = \bigcup_{i=1}^{s} V_{a,b_i} \in \mathcal{J}_{m_a}.$$

Then $a \in W_a$, $B \subseteq V^a$. Moreover, for any $z \in W_a$ and any i, $z \in U_{a,bi}$ and $U_{a,bi} \cap V_{a,bi} = \emptyset$, so $z \notin V_{a,bi}$ for every i, giving $W_a \cap V^a = \emptyset$ and in particular $W_a \cap B = \emptyset$.

Step 2. The family $\{W_a \mid a \in A\}$ is an (ω)open cover of A. By (ω)compactness of A there is a finite subcover $\{W_{a1}, \dots, W_{at}\}$. Define

$U = \bigcup_{j=1}^{t} W_{a_j}$,	$V = \bigcap_{j=1}^{t} V^{a_j}$.

Both U and V are (ω)open, $A \subseteq U$, and $B \subseteq V$. Furthermore,

$$U \cap V = \left(\bigcup_{j=1}^{t} W_{a_j}\right) \cap \left(\bigcap_{j=1}^{t} V^{a_j}\right) \subseteq \bigcup_{j=1}^{t} (W_{a_j} \cap V^{a_j}) = \emptyset.$$

Hence X is (ω)normal. □

Corollary 4.7. *If* $(X, \{\mathcal{J}_n\})$ *is* (ω)*compact and* (ω)*Hausdorff, then* $(X, \mathcal{J}_n)$ *is normal for each* $n \in \mathbb{N}$.

Proof. The proof of Theorem 4.6 applies verbatim with all operations restricted to a single level $\mathcal{J}_n$, since finite intersections and unions within $\mathcal{J}_n$ remain in $\mathcal{J}_n$. □

4.4 Countable paracompactness via Lindel¨of and metacompact conditions

Theorem 4.8. *Let* $(X, \{\mathcal{J}_n\})$ *be simultaneously* (ω)*Lindel¨of and* (ω)*metacompact. If* $(X, \mathcal{J}_n)$ *is normal for each* $n \in \mathbb{N}$ *then* X *is countably* (ω)*paracompact.*

Proof. We verify condition (b) of Theorem 2.1, namely that every countable (ω)open cover has a point-finite ($\mathcal{J}_n$)open refinement for some n.

Let $\{U_i \mid i \in \mathbb{N}\}$ be a countable (ω)open cover of X. Since X is (ω)metacompact, there exist $n \in \mathbb{N}$ and a ($\mathcal{J}_n$)open point-finite refinement $\{W_\beta \mid \beta \in B\}$ of $\{U_i\}$.

By Proposition 3.2 (applied because X is (ω)Lindel¨of), the classical space $(X,\mathcal{J}_n)$ is Lindelo¨f. Hence the $(\mathcal{J}_n)$open cover $\{W_\beta \mid \beta \in B\}$ has a countable subcover $\{W_{\beta j} \mid j \in \mathbb{N}\}$. This subcollection remains point-finite (every subfamily of a point-finite family is point-finite) and still refines $\{U_i\}$. For each $i \in \mathbb{N}$ define

$$Ti = [\{W\beta j \mid W\beta j \subseteq Ui\}.$$

Then:

- Each $T_i \in \mathcal{J}_n$ (a union of $(\mathcal{J}_n)$open sets) and $T_i \subseteq U_i$.
- The family $\{T_i \mid i \in \mathbb{N}\}$ covers X: for any $x \in X$ there exists j with $x \in W_{\beta j}$, and since $\{W_{\beta j}\}$ refines $\{U_i\}$, $W_{\beta j} \subseteq U_{i0}$ for some i_0, giving $x \in T_{i0}$.
- The family $\{T_i\}$ is point-finite: every $x \in X$ belongs to only finitely many $W_{\beta j}$, hence to at most finitely many T_i.

Since $(X,\mathcal{J}_n)$ is normal and $\{T_i \mid i \in \mathbb{N}\}$ is a countable point-finite $(\mathcal{J}_n)$open cover of X, Theorem 2.2 provides a $(\mathcal{J}_n)$open cover $\{V_i \mid i \in \mathbb{N}\}$ of X satisfying $(\mathcal{J}_n)\mathrm{cl}(V_i) \subseteq T_i \subseteq U_i$ for each $i \in \mathbb{N}$. By Theorem 2.1(c), X is countably (ω)paracompact. □

Corollary 4.9. *If $(X,\{\mathcal{J}_n\})$ is (ω)Lindel¨of, (ω)metacompact, (ω)compact, and (ω)Hausdorff, then X is countably (ω)paracompact.*

Proof. Corollary 4.7 gives $(\mathcal{J}_n)$normality for each n, and Theorem 4.8 then applies. □

5 SUMMARY OF IMPLICATIONS

The results of Sections 3 and 4 yield the following diagram of implications, in which each arrow is strict (the examples in Section 3 provide the separating instances):

$$\begin{array}{ccccc} (\omega)\text{compact} & \Rightarrow & (\omega^s)\text{Lindelo¨f} & \Rightarrow & (\omega)\text{Lindelo¨f} \\ \Downarrow & & & & \end{array}$$

$$(\omega)\text{paracompact} \Rightarrow (\omega)\text{metacompact} \Rightarrow \text{countably } (\omega)\text{metacompact}$$

Under additional separation or regularity hypotheses the diagram is enriched by the following implications.

(i) (ω^s)Lindelo¨f + $(\mathcal{J}_n)$regular for all $n \Rightarrow$ (ω)paracompact (Theorem 3.8).
(ii) (ω)Lindel¨of + (ω)metacompact + $(\mathcal{J}_n)$normal for all $n \Rightarrow$ countably (ω)paracompact (Theorem 4.8).
(iii) (ω)compact + (ω)Hausdorff $\Rightarrow$ (ω)normal (Theorem 4.6).
(iv) (ω)Lindel¨of + (ω)compact $\Rightarrow$ product is (ω)Lindelo¨f (Theorem 3.7).
(v) (ω)continuous surjective image of (ω)Lindelo¨f $\Rightarrow$ (ω)Lindelo¨f (Theorem 3.6).

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