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PROPER d-LUCKY LABELING FOR SOME FAMILY OF TREE GRAPHS

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ABSTRACT
In this paper proper d-lucky number is computed for slim tree and Christmas tree. Proper d-lucky number is defined as For a vertex u in a graph G, let $N(u) = \{v \in V(G)/uv \in E(G)$. Let $l : V(G) \rightarrow \{1,2,\dots,k\}$ be a labeling of vertices of a graph G by positive integers. Define $c(u) = \sum_{v\in N(u)} l(v) + d(u)$, where $d(u)$ denotes the degree of u. Define a labeling l as d-lucky if $c(u) \neq c(v)$, for every pair of adjacent vertices u and v in G. The d-lucky number of a graph G, denoted by $\eta_{dl}(G)$, is the least positive k such that G has a d-lucky labeling with $\{1,2,\dots,k\}$ as the set of labels. A d-lucky labeling is said to be proper d-lucky labeling if for every pair of adjacent vertices u and v in G, $u \neq v$, and is denoted by $\eta_{pdl}(G)$.

KEYWORDS- ***slim tree, Christmas tree, lucky labeling proper lucky labeling, d-lucky number, proper d-lucky number.***

1. INTRODUCTION

Labeling of graph is an area where a lot of research is under progress. There are so many variants in labeling of graphs. Among the many variants, Lucky and Proper Lucky Labeling of Quadrilateral Snake Graphs were studied by T. V. Sateesh Kumar et al [1]. Recently a new variant of labeling was introduced as d- Lucky labeling by Mirka Miller et al [2]. d-lucky number, its lower bound and several exact results were found by Sandi Klavzar et al [3]. For Honeycomb network d-lucky labeling was shown by A. Sahayamary et al [4]. Proper lucky number for certain rooted product graphs and proper d-lucky labeling of rooted products and corona products of certain graphs were studied Kujur C[5,6].

For a vertex u in a graph G, let $\mathrm{N(u)} = \{\mathrm{v} \in \mathrm{V(G)/uv} \in \mathrm{E(G)}$. Let $\mathrm{l} : \mathrm{V(G)} \rightarrow \{1,2,\dots,\mathrm{k}\}$ be a labeling of vertices of a graph G by positive integers. Define $c(\mathrm{u}) = \sum_{\mathrm{v}\in \mathrm{N(u)}} \mathrm{l(v)} + \mathrm{d(u)}$, where $\mathrm{d(u)}$ denotes the degree of u. Define a labeling l as d-lucky if $c(\mathrm{u}) \neq \mathrm{c(v)}$, for every pair of adjacent vertices u and v in G. The d-lucky number of a graph G, denoted by $\eta_{\mathrm{dl}}(\mathrm{G})$, is the least positive k such that G has a d-lucky labeling with $\{1,2,\dots,\mathrm{k}\}$ as the set of labels [2]. For simplification let us name $c(u)$ as cumulative sums. A d-lucky labeling is said to be proper d-lucky labeling if for every pair of adjacent vertices u and v in G, $u \neq v$, and is denoted by $\eta_{pdl}(G)$.

2. PROPER d-LUCKY LABELING OF SLIM TREE

Slim tree was defined by C.N. Hung et al. [7] as: For $s \geq 2$, the s^{th} slim tree $ST(s) = (V, E, u, l, r)$, where V the node set, E the edge set, $u \in V$ the root node, $l \in V$ the left node and $r \in V$ the right node, is recursively defined as follows:

1. $ST(2)$ is the complete graph K_3 with its nodes labeled with u, l and r.
2. The s^{th} slim tree $ST(s)$, with $s \geq 3$ is composed of a root node u and two disjoint copies of $(s-1)^{th}$ slim trees as the left subtree and the right subtree, denoted by $ST^l(s-1) = (V_1, E_1, u_1, l_1, r_1)$ and $ST^r(s-1) = (V_2, E_2, u_2, l_2, r_2)$ respectively and $ST(s) = (V, E, u, l, r)$ is given by $V = V_1 \cup V_2 \cup \{u\}$, $E = E_1 \cup E_2 \cup \{(u,u_1),(u,u_2),(r_1,l_2)$, $l = l_1$, $r = r_2$. The labeling of a slim tree is given below in the fig.1, we name V_1^0 as root vertex and other vertex are in the levels as 1st, 2nd, …..and n^{th}(which is also called the base level).

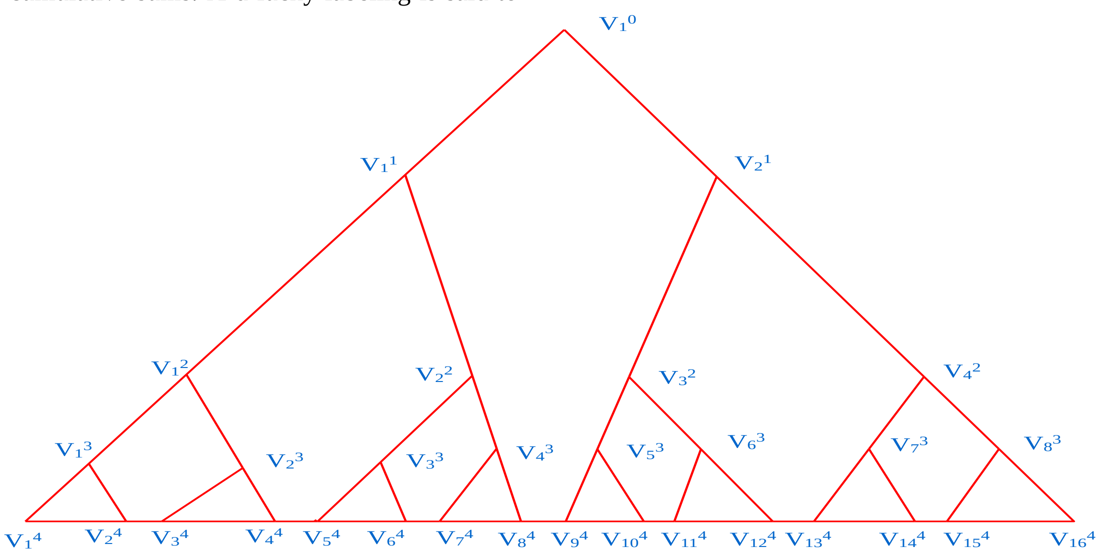

Figure 1. Labeling of slim tree

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Theorem 2.1: A slim Tree $ST(s)$ admits Proper d-Lucky labeling and $\eta_{pdl}(ST(s)) \leq 4$.

Proof:

Label the root vertex V_1^0 as 2 the vertices of $ST^l(s-1)$ and $ST^r(s-1)$ are labeled as given below:- The vertices (V_1^1, V_2^1) receive the as 3 and the vertices $(V_1^2, V_2^2, V_3^2, V_4^2)$ get the label as 1. The process is repeated till $(V_1^{n-1}, V_2^{n-1}, \dots, V_{2^{n-1}}^n)$ vertices. The vertices in the base $(V_1^n, V_2^n, \dots, V_{2^n}^n)$ are labeled as follows: $V_1^n = 4$, and $V_{2^n}^n = 4$, rest of the vertices are labeled as 1, 2 like $V_2^n = 1$, $V_3^n = 2$, $V_4^n = 1, \dots V_{2^n-1}^n = 2$ alternately, if the vertices above the base vertices have the label 3 and if the vertices above the base vertices have the label as 1 then the base vertices are labeled as 2, 3 alternately. Two cases arise for the neighbourhood sum $s(u)$ and cumulative sum $c(u)$:-

<u>Case 1</u>:

If the vertices $(V_1^{n-1}, V_2^{n-1}, \dots, V_{2^{n-1}}^{n-1})$ are labeled as 3, then the neighbourhood sum of the root vertex is $s(V_1^0) = 6$ and the cumulative sum is $c(u) = 8$. All the vertices, labeled as 3 above the base vertices $(V_1^n, V_2^n, \dots, V_{2^n}^n)$ have the neighbourhood sums as $s(u) = 4$ and cumulative sums as $c(u) = 6$, except the end vertices which have $s(V_1^{n-1}) = 6$ and $c(V_1^{n-1}) = 9$, $s\left(V_{2^{n-1}}^{n-1}\right) = 7$ and $c\left(V_{2^{n-1}}^{n-1}\right) = 10$. The vertices which are labeled as 1 above the base vertices have the neighbourhood sum as 9 and cumulative sum as 12. The base V_1^n has the neighbourhood sum as 4 and cumulative sum as 6, V_2^n has the neighbourhood sum as 9 and cumulative sum as 12, the vertex $V_{2^n-1}^{n-1}$ has the neighbourhood sum as 8 and cumulative sum as 11, the vertex $V_{2^n}^n$ has the neighourhood sum as 5 and cumulative sum as 7. All the remaining vertices, with the label as 2 have the neoighbourhood sums as 5 and cumulative sums as 8. The vertices which as labeled as 1 have the neighbourhood sums as 7 and cumulative sums as 10. It can be seen that no two adjacent vertices have the same neighbourhood and cumulative sums. (For illustration see Figure 2)

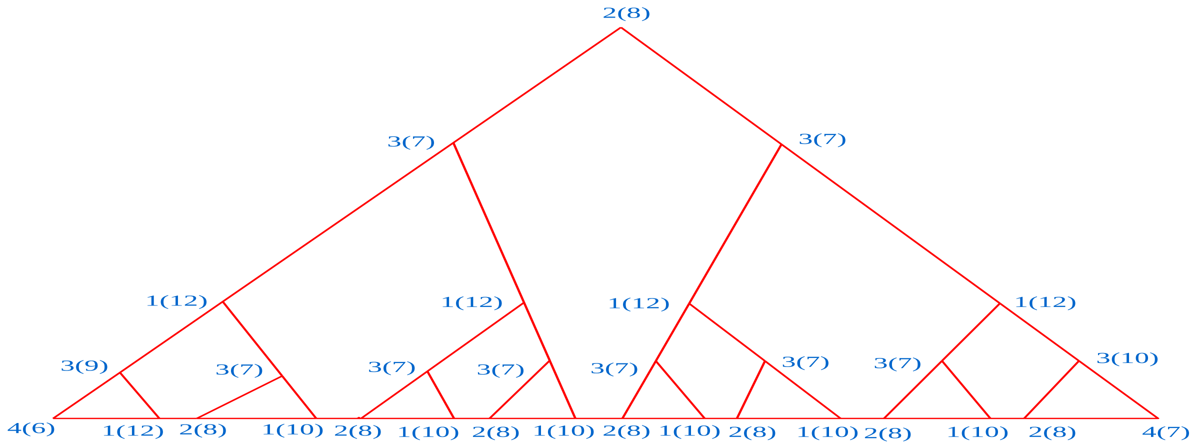

Figure 2 proper d-lucky labeling of ST(4)

Case 2:

If the vertices $(V_1^{n-1}, V_2^{n-1}, \dots, V_{2^{n-1}}^{n-1})$ are labeled as 1, then the root vertex has the neighbourhood sum as $s(V_1^0) = 6$ and the cumulative sum as $c(V_1^0) = 8$. All the vertices, labeled as 3 at level 1 i.e. $V_1^1, V_2^1, \dots, V_n^1$ have the neighbourhood sums as 4 and cumulative sums as 6. All other vertices with label as 3 except at the base vertices have the neighbourhood sum as 3 and cumulative sums as 5. All the vertices, labeled as 1 except at the level $(n-1)$ have the neighbourhood sum as 9 and cumulative sums as 12. The vertices at $(n-1)^{th}$ level which are labeled as 1 have the neighbourhood sums as 8 and cumulative sums as 11. At the base of the slim tree the vertex V_1^n has the neighbourhood sum as 4 and the cumulative sum as 6. The vertex V_n^n has the neighbourhood sum as 3 and cumulative sum as 5. All the vertices at base label as 2 have the neighbourhood sums as 7 and cumulative sums as 10, the vertices, labeled as 3 have the neighbourhood sums as 5 and cumulative sums as 8. (For illustration see Figure 3)

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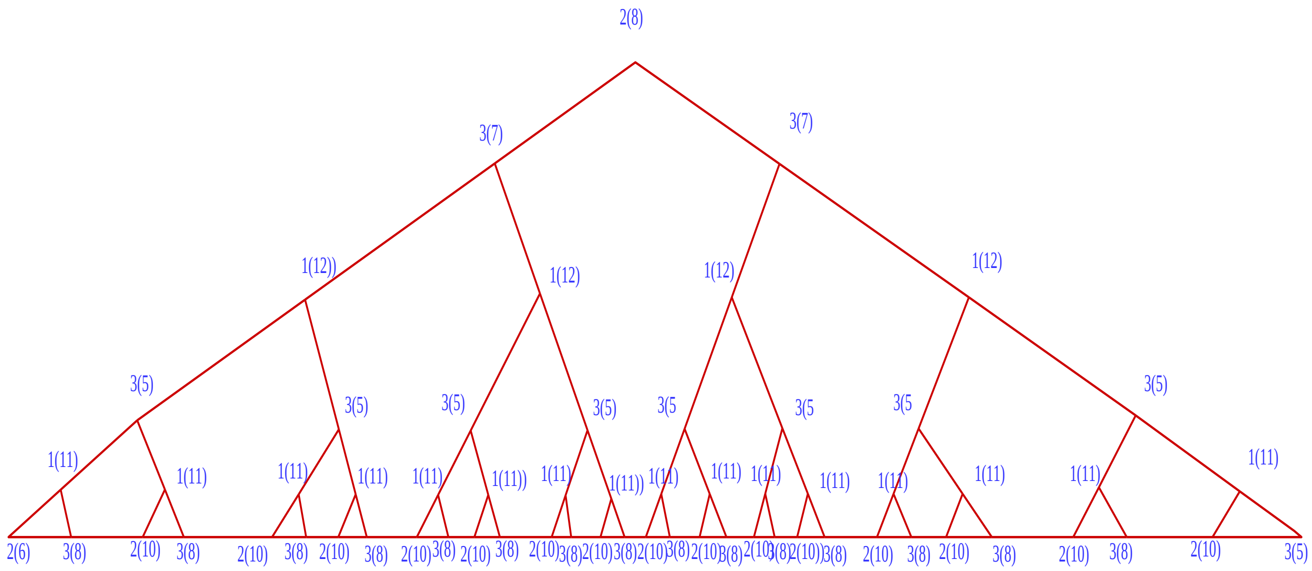

Figure 3 Proper d-lucky labelling of ST(5)

It is seen no two adjacent vertices have the same neighbourhood sums and cumulative sums. Hence a slim Tree $ST(s)$ admits proper d-lucky labeling and $\eta_{pdl}(ST(s)) \leq 4$.

3. PROPER D-LUCKY LABELING OF CHRISTMAS TREE

Christmas tree was defined by C.N. Hung et al. [7] as: For $s \geq 2$, the Christmas tree $CT(s)$ is composed of an s^{th} slim tree $ST(s) = (V_1, E_1, u_1, l_1, r_1)$ and an $(s+1)^{th}$ slim tree $ST(s+1) = (V_2, E_2, u_2, l_2, r_2)$ together with the edges $(u_1, u_2), (l_1, r_2)$and (l_2, r_1). The labeling of Christmas tree is shown below in the figure 4, where the vertex V_1^0 is known as root vertex and other vertices are at 1st , 2nd , …, n^{th} level. The vertices at n^{th} levels are also known as base vertices.

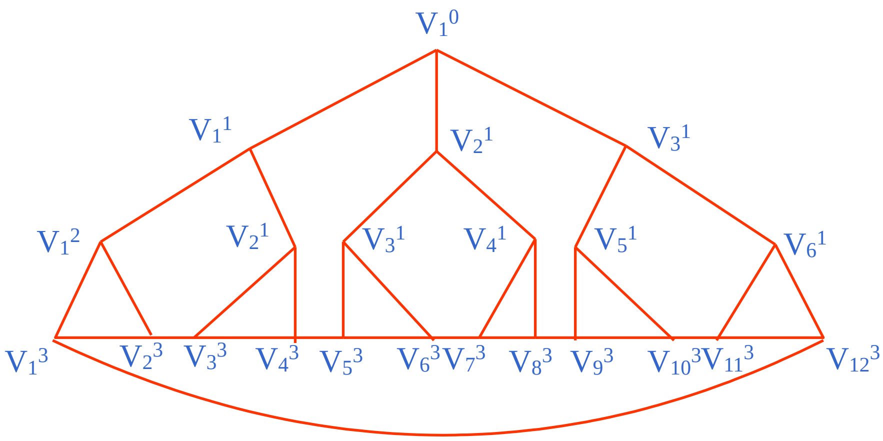

Figure 4 labelling of CT(4)

Theorem 3.1: A Christmas Tree $CT(s)$ admits Proper d-lucky labeling and $\eta_{pdl}(CT(s)) \leq 4$.

Proof:

Label the vertices of the Christmas trees as given below:

Label the root vertex V_1^0 as 2 and label the vertices (V_1^1, V_2^1, V_3^1) as 1. Label the vertices $(V_1^2, V_2^2, V_3^2, V_4^2, V_5^2, V_6^2)$ as 3. Repeat the process up to $(V_1^{n-1}, V_2^{n-1}, \ldots, V_{3.2^{n-1}}^{n-1})$ vertices with labels as 1 and 3. Label the base vertices $(V_1^n, V_2^n, \ldots, V_{3.2^n}^n)$ as $V_1^n = 1$, $V_2^n = 4$, and $V_{3.2^n}^n = 4$. Label other vertices in the base as $V_3^n = 1, V_4^n = 2, \ldots V_{3.2^n-1}^n = 1$ alternatively, If the vertices $(V_1^{n-1}, V_2^{n-1}, \ldots, V_{3.2^{n-1}}^{n-1})$ are labeled 3. If the vertices $(V_1^{n-1}, V_2^{n-1}, \ldots, V_{3.2^{n-1}}^{n-1})$ are labeled as 1 then the vertices in the base labeled as $V_1^n = 2, V_2^n = 3, V_3^n = 2, V_4^n = 3, \ldots, V_{3.2^n-1}^n = 2$ and $V_{3.2^n}^n = 3$ alternatively. There are two cases while calculating the neighborhood and cumulative sums: -

Case 1:

If the vertices above the base get label as 3, then the root vertex V_1^0 has the neighbourhood sum as $s(V_1^0) = 3$ and $c(V_1^0) = 6$.

The vertices at level 1 which are labeled as 1 have the neighbourhood sums as 8 and cumulative sums as 11. All the remaining vertices which are labeled as 1 above the base have the nieghbourhood sums as 9 and cumulative sums as 12. The vertices above the $(n-1)^{th}$ level with label as 3 have neighbourhood sums as 3 and cumulative sums as 6. The vertices at the $(n-1)^{th}$ level have the following neighbourhood and cumulative sums $s(V_1^{n-1})=6$, c$((V_1^{n-1})=9$, $s\left(,V_{3.2^{n-1}}^{n-1}\right)=6$, $c\left(,V_{3.2^{n-1}}^{n-1}\right)=9$ and the remaining vertices have the neighbourhood sums as 4 and cumulative sums as 7. At the n^{th} level or at the base of the Christmas tree the neighbourhood sums and cumulative sums are as follows:- $s(V_1^n)=11$, $c(V_1^n)=14, s(V_2^n)=5$, $s(V_2^n)=8$, $s(V_3^n)=9$, $c(V_3^n)=12$, $s\left(V_{3.2^n-1}^n\right)=9$, $s\left(V_{3.2^n-1}^n\right)=12$, $s\left(V_{3.2^n}^n\right)=5$ and c$\left(V_{3.2^n}^n\right)=8$. The vertices which are labeled as 1 have the neighbourhood sums as 7 and cumulative sums as 10, the vertices which are labeled as 2 have the neighbourhood sums as 5 and cumulative sums as 8. It is observed that no two adjacent vertices have the same neighbourhood and cumulative sums. (for illustration see Figure 6)

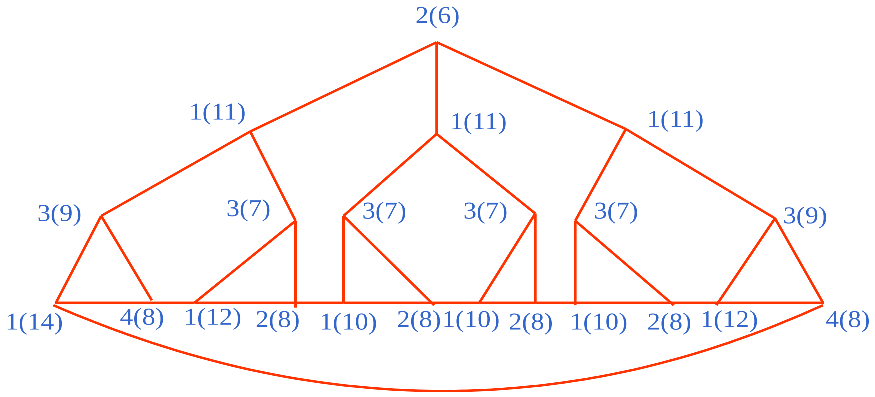

Figure 6 Proper d-lucky labeling of CT(3)

***Case 2*:**

If the vertices above the base of the Christmas tree are labeled as 1, then the root vertex V_1^0 has he neighbourhood sum as $s(V_1^0)=3$ and $s(V_1^0)=6$. The vertices at level 1 and at $(n-1)^{th}$ level which are labeled as 1 have the neighbourhood sums as 8 and cumulative sums as 11. The remaining vertices between level 1 and level $(n-1)$ which are labeled as have the nieghbourhood sums as 9 and cumulative sums as 12. All vertices above n^{th} level of Christmas tree labeled as 3 have neighbourhood sums as 3 and cumulative sums as 6. At the base all the vertices labeled as 2 have the neighbourhood sums as 7 and cumulative sums as 10, the vertices which are labeled as 3 have the neighbourhood sums as 5 and cumulative sums as 8. (for illustration see figure 7)

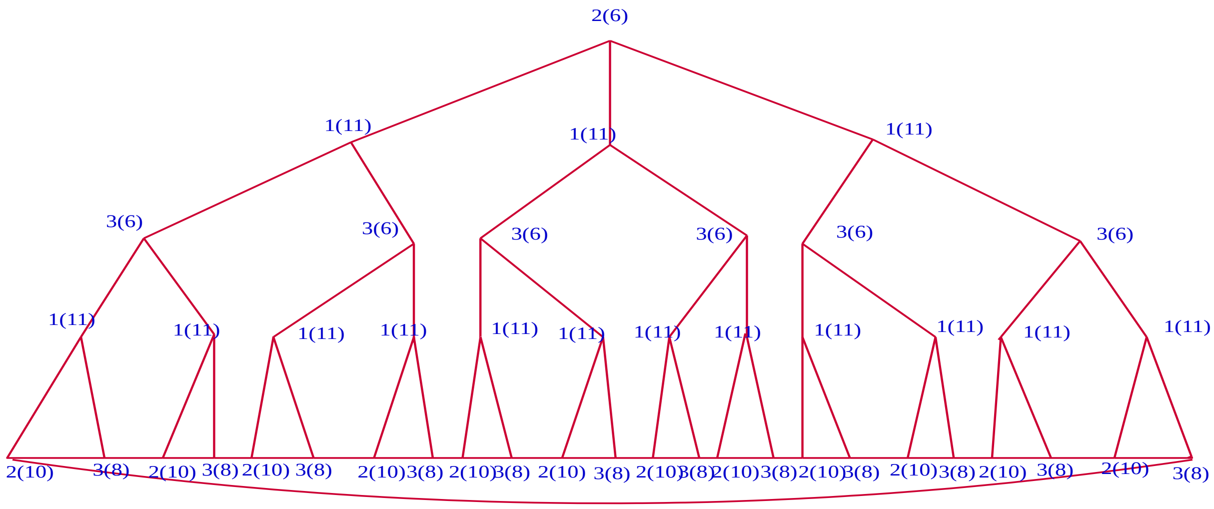

Figure 7 proper d-lucky labeling of CT(4)

In both the cases it is seen that no two adjacent vertices have the same neighbourhood sums and cumulative sums. Thus, a Christmas Tree $CT(s)$ admits proper d-lucky labeling and $\eta_{pdl}(CT(s))\leq 4$.

4. CONCLUSION

In this paper proper d-lucky numbers were computed for slim tree, Christmas tree and found as $\eta_{pdl}(ST(s))\leq \mathbf{4}$, $\eta_{pdl}(\boldsymbol{CT}(\boldsymbol{s}))\leq \mathbf{4}$.

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