ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 7| July 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

ON FINDING INTEGER SOLUTIONS TO QUATERNARY QUADRATIC DIOPHANTINE EQUATION $xy+12zw=3(x+y)(z+w)$

N.Thiruniraiselvi[1*], M.A.Gopalan[2], Sharadha Kumar[3]

[1]Associate Professor, Department of Mathematics, Vignesh College of Engineering and Technology, (Affiliated to Anna University, Chennai), Koothur, Trichy- 621 216,Tamil Nadu, India.*
https://orcid.org/0000-0003-4652-3846
[2]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Tiruchirappalli-620002, Tamil Nadu, India.
https://orcid.org/0000-0003-1307-2348
[3]Teacher,Department of Mathematics, National Public School, Jakkur, Bangalore-560064, India
https://orcid.org/0000-0002-0509-6158

ABSTRACT

This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by $xy+12zw=3(x+y)(z+w)$*.Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduce the quadratic equation with four unknowns to solvable ternary quadratic equation of the form* $z^2=Dx^2+y^2$*,* $D>0$ *and square free. .*

KEYWORDS *: Quaternary quadratic equation, Homogeneous quadratic equation, Integer solutions*

Notations

$$t_{3,n}=\frac{n(n+1)}{2}$$

$$P_n^3=\frac{n(n+1)(n+2)}{6}$$

$$P_n^5=\frac{n^2(n+1)}{2}$$

INTRODUCTION

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety .In particular, quadratic Diophantine equations have a wealth of historical significance. In this context, one may refer [1-21] for second degree diophantine equations with many unknowns. . This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by $xy+12zw=3(x+y)(z+w)$.Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduce the quadratic equation with four unknowns to solvable ternary quadratic equation of the form $z^2=Dx^2+y^2$, $D>0$ and square free .

Method of analysis

The polynomial equation of second degree with four unknowns to be solved is

$$.xy+12zw=3(x+y)(z+w) \qquad (1)$$

The substitution of the linear transformations

$$x = u + p, y = u - p, z = u + q, w = u - q, u \neq \pm p, q \qquad (2)$$

in (1) leads to the ternary quadratic equation

$$u^2 = p^2 + 12q^2 \qquad (3)$$

which is satisfied by

$$q = 2rs, p = 12r^2 - s^2, u = 12r^2 + s^2$$

From (2), the integer solutions to (1) are given by

$$x = 24r^2, y = 2s^2, z = 12r^2 + s^2 + 2rs, w = 12r^2 + s^2 - 2rs \qquad (4)$$

Note 1

It is seen that ,by expressing (3) as the system of double equations ,the following seven patterns of integer solutions to (1) are obtained:

Pattern 1

$$x = 8s, y = 6s, z = 9s, w = 5s$$

Pattern 2

$$x = 24s, y = 2s, z = 15s, w = 11s$$

Pattern 3

$$x = 4s^2, y = 12, z = 2s^2 + 2s + 6, w = 2s^2 - 2s + 6.$$

Pattern 4

$$x = 2q^2, y = 6, z = q^2 + q + 3, w = q^2 - q + 3.$$

Pattern 5

$$x = 12s^2, y = 4, z = 6s^2 + 2s + 2, w = .6s^2 - 2s + 2$$

Pattern 6

$$x = 6q^2, y = 2, z = 3q^2 + q + 1, w = 3q^2 - q + 1.$$

Pattern 7

$$x = 6s, y = 2, z = 4s + 1, w = 2s + 1$$

The procedure to obtain other patterns of integer solutions to (1) through

solving (3) is as below:

Procedure 1

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 7| July 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

Consider (3) as

$$u^2 - 12\,q^2 = p^2 * 1 \tag{5}$$

Assume

$$p = a^2 - 6b^2 \tag{6}$$

The integer 1 in (5) is written as

$$1 = (7 + 2\sqrt{12})(7 - 2\sqrt{12}) \tag{7}$$

Substituting (6) & (7) in (5) and using factorization, we have

$$u + \sqrt{12}\,q = (7 + 2\sqrt{12})(a + \sqrt{12}\,b)^2$$

from which, we get

$$u = 7(a^2 + 12b^2) + 48ab,$$
$$q = 2(a^2 + 12b^2) + 14ab.$$

In view of (2) , the integer solutions to (1) are given by

$$\left.\begin{aligned} x &= 8a^2 + 72b^2 + 48ab \\ y &= 6a^2 + 96b^2 + 48ab \\ z &= 9(a^2 + 12b^2) + 62ab \\ w &= 5(a^2 + 12b^2) + 34ab \end{aligned}\right\} \tag{8}$$

A few numerical examples from (8) are given in Table-1 below:

Table-1 Numerical Examples

a	B	X	y	z	W
1	1	128	150	179	99
2	1	200	216	268	148
1	2	392	486	565	313
4	2	800	864	1072	592

Observations

1. Each of the following expressions is a perfect square when a=b

$$\left.\begin{aligned} &5z - 9w \\ &7(z - w) - 2(x + y) \\ &9(z - w) - 4z \\ &5(x + y) - 14w \\ &14z - 9(x + y) \end{aligned}\right\} \tag{9}$$

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 7| July 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

2. When a=b+1 , each of the expressions in (9) = $8\,t_{3,b}$
3. When a=2 b , each of the expressions in (9) is a square multiple of 8
4. When a= (b+1) (b+2) , each of the expressions in (9) = $24\,P_b^3$
5. When a=b (b+1) , each of the expressions in (9) = $8\,P_b^5$
6. When b=1 , $3(x-y+24)$ is a nasty number

Note 2

Apart from (7), the integer 1 in (5) is written as

$$1=\frac{(4+\sqrt{12})\,(4-\sqrt{12})}{4}$$

$$1=\frac{(12\,r^2+s^2+2rs\sqrt{12})\,(12\,r^2+s^2-2rs\sqrt{12})}{(12\,r^2-s^2)^2}$$

Following the above procedure, two more sets of integer solutions to (1) are obtained.

Procedure 2

Rewrite (3) as

$$p^2+12\;q^2=u^2*1 \tag{10}$$

Assume

$$u=(4s^2+3)^2\;(a^2+12\,b^2) \tag{11}$$

Express the integer 1 in (10) as

$$1=\frac{(4s^2-3+i2s\sqrt{12})\,(.4s^2-3-i2s\sqrt{12})}{(4s^2+3)^2} \tag{12}$$

Substituting (11) & (12) in (10) and using factorization, we have

$$p+i\sqrt{12}\,q=(4s^2+3)\,(4s^2-3+i\,2s\sqrt{12}\,)\,(a+i\sqrt{12}\,b)^2$$

from which we get

$$p=(4s^2+3)\,[(4s^2-3)\,(a^2-12b^2)-48\,sab]$$
$$.q=(4s^2+3)\,[2s\,(a^2-12b^2)+2\,ab(4s^2-3)] \tag{13}$$
.

Substituting (11) & (13) in (2), the required integer solutions to (1) are obtained.

Note 3

It is to be noted that the integer 1 in (10) may be considered as

$$1=\frac{(12r^2-s^2+i2rs\sqrt{12})\,(12r^2-s^2-i2rs\sqrt{12})}{(12r^2+s^2)^2}$$

$$1=\frac{(4s^2-4s-2+i(2s-1)\sqrt{12})\,(.4s^2-4s-2-i(2s-1)\sqrt{12})}{(4s^2-4s+4)^2}$$

Repeating the above process and taking different values to r& s, one obtains different sets of integer solutions to (1).

Procedure 3

Introduction of the transformations

$$x=.3y,.z=p+q,w=p-q\ ,p\neq\pm q \tag{14}$$

in (1) leads to the equation

$$y^2-.8py+4(p^2-q^2)=0$$

Treating the above equation as a quadratic equation in y and solving for the same ,we have

$$y=4p\pm2\sqrt{q^2+3p^2} \tag{15}$$

The square-root on the R.H.S. of (15) is removed when

$$p=2rs,q=3r^2-s^2 \tag{16}$$

Substituting (16) in (15) & (14) , we have the integer solutions to (1) given by

$$y=.2[4rs\pm(3r^2+s^2)]$$
$$x=.6[4rs\pm(3sr^2+s^2)]$$
$$z=2rs+3r^2-s^2$$
$$w=2rs-3r^2+s^2$$

Note 4

The square-root on the R.H.S. of (15) is also removed when

(i) $p=2t+1,\ q=2t^2+2t-1$

(ii) $p=2t+1,q=6t^2+6t+1$

we, after some algebra, obtain the respective integer solutions to (1) to be

$$y=4(2t+1)\pm2(2t^2+2t+2),x=.3[4(2t+1)\pm2(2t^2+2t+2)],z=2t^2+4t,w=-2\,t^2+2$$
$$y=4(2t+1)\pm2(6t^2+6t+2),x=.3[4(2t+1)\pm2(6t^2+6t+2)],z=6t^2+8t+2,w=-6\,t^2-4t$$

Conclusion

Varieties of solutions in integers are presented in this paper for the quaternary homogeneous quadratic equation given by $xy+12zw=3(x+y)(z+w)$ through employing substitution technique and factorization method. As quadratic equations (homogeneous or non-homogeneous) are plenty, one may search for patterns of integer solutions to other choices of multivariable quadratic equations.

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 7| July 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

REFERENCES

1. *M.A.Gopalan, S.Vidhyalakshmi andT.R.Usharani, Integral Points on the Non- homogeneous Cone* $2z^2+4xy+8x-4z+2=0$ *, Global Journal of Mathematics and Mathematical Sciences, Vol. 2(1), Pp. 61-67, 2012.*
2. *M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, Observations on the ternary quadratic Diophantine equation* $x^2+9y^2=50z^2$ *, International Journal of Applied Research, Vol.1 (2), Pp.51-53, 2015.*
3. *M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, Construction of Diophantine quadruple through the integer solution of ternary quadratic Diophantine equation* $x^2+y^2=z^2+4n$ *, International Journal of Innovative Research in Engineering and Science,Vol.5(4), Pp.1-7, May-2015*
4. *M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, Observations on the cone* $z^2=ax^2+a(a-1)y^2$ *, International Journal of Multidisciplinary Research and Development, Vol.2 (9), Pp.304-305, Sep-2015.·*
5. *N.Thiruniraiselvi, M.A.Gopalan, On the equations* $y^2=11x^2+22$ *, Academic Journal of Applied Mathematical Sciences AJAMS, Vol 6(7), Pp.85-92, July 2020.*
6. *M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, A Study on Special Homogeneous Cone* $z^2=24x^2+y^2$ *,Vidyabharati International Interdisciplinary Research Journal, (Special 3Issue on "Recent Research Trends in Management, Science and Technology"-Part-3, pdf page-330), Pg: 1203-1208, 2021.*
7. *M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, On the Homogeneous Ternary Quadratic Diophantine Equation* $6x^2+5y^2=341z^2$ *,Vidyabharati International Interdisciplinary Research Journal, (Special Issue on "Recent Research Trends in Management, Science and Technology"-Part-4, pdf page- 318), Pg: 1612-1617, 2021.*
8. *M.A.Gopala,S.Vidhyalakshmi, J.Shanthi, V.Anbuvalli, On Finding the integer solutions of Ternary Quadratic Diophantine Equation* $3(x^2+y^2)-5xy=36z^2$ *, International Journal of Precious Engineering Research and Application(IJPERA), volume 7,Issue 1 , 34-38, may 2022.*
9. *J.Shanthi, M. Parkavi "On Finding Integer Solutions To The Homogeneous Ternary*
   a. *Quadratic Diophantine Equation* $2(x^2+y^2)-3xy=32z^2$ *" International Journal of Research Publication and Reviews, Vol 4, no 1, pp 700-708,January 2023.*
10. *J.Shanthi, V. Anbuvalli, M.A.Gopalan,S. Vidhyalakshmi, On finding integer solutions to the homogeneous cone* $x^2+(k^2+2k)y^2=(k+1)^4z^2$ *, International Journal of Mathematics and computer Research, Volume 11, Issue 07, Pp 3555-3557, July 2023.*
11. *M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, A class of new solutions in integers to Ternary Quadratic Diophantine Equation* $12(x^2+y^2)-23xy+2x+2y+4=56z^2$ *,International Journal of Research Publication and Reviews, Vol 5 (5), Page – 3224-3226,May (2024).*
12. *J.Shanthi,M.A.Gopalan, A Glimpse on Homogeneous Ternary Quadratic Diophantine Equation* $39(x^2+y^2)+72xy=246z^2$ *,IARJSET,11(9),96-101,2024*
13. *N.Thiruniraiselvi, M.A.Gopalan, A Glimpse on Finding Integer Solutions to Homogeneous Quaternary Quadratic Diophantine Equation* $xy-2zw+(z-w)^2=(x+y)(z+w)$ *International Journal of Advance Research Publications, 2(5), 01-09, 2026.*
14. *N.Thiruniraiselvi, M.A.Gopalan, On Finding Integer Solutions to Homogeneous Ternary Quadratic Diophantine Equation* $x^2+(2k+1)y^2=(k+1)^2z^2$*, Rattanakosin Journal of Science and Technology: RJST, Volume 7Issue 3: 252-258, 2025.*
15. *R.Sathiyapriya, N.Thiruniraiselvi, M.A.Gopalan, Lattice points on the Cone* $y^2+5x^2=86z^2$ *, Nanotechnology Perceptions, 20(7), 890–898, 2024.*
16. *R.Sathiyapriya, N.Thiruniraiselvi, M.A.Gopalan, A Glimpse on Homogeneous Ternary Quadratic* $3x^2+2y^2=275z^2$*, Journal of Computational Analysis and Applications, 33(7),693-696, 2024*
17. *N.Thiruniraiselvi, M.A.Gopalan, On Finding Integer Solutions To Homogeneous Ternary Quadratic Diophantine Equation* $x^2+(2k+1)y^2=(k+1)^2z^2$ *, International Journal of Recent Scientific Research Vol. 15, Issue, 09, pp.5003-5005, September, 2024.*
18. *N.Thiruniraiselvi, Sharadha Kumar, M.A.Gopalan, Second Degree Quinary Equation* $xy+XY=(k^2-2k+1)w^2$ *,"Advances in Nonlinear Variational Inequalities", 27(2), 18-22, 2024.*
19. *N.Thiruniraiselvi, M.A.Gopalan,Technique on Solving a Binary Quadratic Diophantine Equation Involving Odd Powers of 23, International Journal of Research in Academic World, Vol 3(7), 196-198, (July 2024).*

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 7| July 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

20. *N.Thiruniraiselvi, M.A.Gopalan, Technique on Solving a Binary Quadratic Diophantine Equation* $3x^2 + 5y^2 = 17^{(2k+1)}$*, Indiana Journal of Multidisciplinary Research, 4(4): 1-3, 2024.*
21. *N.Thiruniraiselvi, M.A.Gopalan, New Class of Integer Solutions to Ternary Quadratic Equation* $x^2 + y^2 = (a^2 + b^2)z^2$*, Arhimede Mathematical Journal,Vol.11(1), 21-25, 2024.*