ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 9| September 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

TECHNIQUES ON SOLVING NON-HOMOGENEOUS TERNARY SEXTIC SURFACE

$x^2 - xy + y^2 = 19z^6$

V.Anbuvalli[1], J. Shanthi[2], M.A. Gopalan[3]

[1] *Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Tiruchirappalli-620 002, Tamil Nadu, India.*
https://orcid.org/0009-0006-0905-9123

[2]*Assistant Professor and Head , Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Tiruchirappalli-620002, Tamil Nadu, India.*
https://orcid.org/0009-0008-5945-410X

[3]*Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Tiruchirappalli-620002, Tamil Nadu, India.*
https://orcid.org/0000-0003-1307-2348

ABSTRACT

The main aim of this paper is to illustrate the process of obtaining non-zero distinct integer solutions to the non-homogeneous ternary Sextic equation given by $x^2 - xy + y^2 = 19z^6$ *. Substitution technique and factorization method are utilized to obtain the same. We are able to solve the given Sextic equation through the methods suggested above for obtaining plenty of non-zero distinct integer solutions.*

KEYWORDS: *Non-homogeneous equation, Ternary sextic equation, Integer Solutions, Substitution technique, Factorization method*

INTRODUCTION

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Diophantine equations with multi variables [1-19]. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree six with three unknowns given by $x^2 - xy + y^2 = 19z^6$.

Method of analysis

The non-homogeneous polynomial equation of degree six with four unknowns under consideration is given by

$$x^2 - xy + y^2 = 19z^6 \quad (1)$$

The insertion of the transformation

$$x = u + v, y = u - v, u \neq v \neq 0 \quad (2)$$

in (1) leads to the non-homogeneous ternary sextic equation

$$u^2 + 3v^2 = 19z^6 \quad (3)$$

To start with, it is seen after some algebra that (3) is satisfied by

$$u = 4\alpha^3 , v = \alpha^3 , z = \alpha$$

and from (2) ,the corresponding values of x,y satisfying (1) are given by

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 9| September 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

$$x = 5\alpha^3$$
$$y = 3\alpha^3$$

The process of obtaining varieties of integer solutions to (1) through solving (3) is presented below:

Process 1

The choice

$$v = k z^2 \quad (4)$$

in (3) gives

$$u^2 = z^4 (19 z^2 - 3k^2) \quad (5)$$

Let

$$\alpha^2 = 19 z^2 - 3k^2 \quad (6)$$

which is the well-known pellian equation whose smallest positive integer solution is

$$z_0 = k,\ \alpha_0 = 4k$$

The general solution (z_{n+1}, α_{n+1}) to (6) is given by

$$z_{n+1} = \frac{k}{2\sqrt{19}}[\sqrt{19}\, f_n + 4 g_n]$$

$$\alpha_{n+1} = \frac{k}{2}[4 f_n + \sqrt{19}\, g_n] \quad (7)$$

where

$$f_n = (170 + 39\sqrt{19})^{n+1} + (170 - 39\sqrt{19})^{n+1}$$
$$g_n = (170 + 39\sqrt{19})^{n+1} - (170 - 39\sqrt{19})^{n+1} \quad (8)$$

From (5) and (4) , one has

$$u_{n+1} = z_{n+1}^{\ 2} * \alpha_{n+1}$$
$$v_{n+1} = k z_{n+1}^{\ 2}$$

In view of (2), we have

$$x_{n+1} = z_{n+1}^{\ 2} * (\alpha_{n+1} + k)$$
$$y_{n+1} = z_{n+1}^{\ 2} * (\alpha_{n+1} - k) \quad (9)$$

Thus, the values of x, y, z given by (9) & (7) satisfy (1).

Process 2

The choice

$$u = 4k z^2 \quad (10)$$

in (3) gives

$$3v^2 = z^4 (19 z^2 - 16 k^2) \quad (11)$$

Let

$$3\alpha^2 = 19 z^2 - 16 k^2 \quad (12)$$

Choosing

$$z = X + 3T, \alpha = X + 19T \quad (13)$$

in (12) , one has

$$X^2 = 57 T^2 + k^2 \quad (14)$$

which is the well-known pellian equation whose smallest positive integer solution is

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 9| September 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

$$T_0 = 20k, X_0 = 151k$$

The general solution (T_{n+1}, X_{n+1}) to (14) is given by

$$T_{n+1} = \frac{k}{2\sqrt{57}}[20\sqrt{57}\, f_n + 151\, g_n]$$

$$X_{n+1} = \frac{k}{2}[151 f_n + 20\sqrt{57}\, g_n] \quad (15)$$ where

$$f_n = (151 + 20\sqrt{57})^{n+1} + (151 - 20\sqrt{57})^{n+1}$$

$$g_n = (151 + 20\sqrt{57})^{n+1} - (151 - 20\sqrt{57})^{n+1} \quad (16)$$

From (11) and (10) , one has

$$v_{n+1} = (z_{n+1})^2 * \alpha_{n+1}$$

$$u_{n+1} = 4k(z_{n+1})^2$$

where

$$z_{n+1} = X_{n+1} + 3T_{n+1}$$

$$\alpha_{n+1} = X_{n+1} + 19T_{n+1} \quad (17)$$ where X_{n+1}, T_{n+1} are given by (15).

In view of (2), we have

$$x_{n+1} = (z_{n+1})^2 * (\alpha_{n+1} + 4k)$$

$$y_{n+1} = (z_{n+1})^2 * (-\alpha_{n+1} + 4k) \quad (18)$$

Thus, the values of x, y, z given by (18) & (17) satisfy (1).

Process 3

Let

$$z = a^2 + 3b^2 \quad (19)$$

Consider the integer 19 on the R.H.S. of (3) as

$$19 = (4 + i\sqrt{3})(4 - i\sqrt{3}) \quad (20)$$

Substitute (19) & (20) in (3) and employ the method of factorization. On equating the positive factors on both sides, one obtains

$$u + i\sqrt{3}\, v = (4 + i\sqrt{3})(a + i\sqrt{3}\, b)^6$$

$$= (4 + i\sqrt{3})[f(a,b) + i\sqrt{3}\, g(a,b)] \quad (21)$$ where

$$f(a,b) = a^6 - 45a^4b^2 + 135a^2b^4 - 27b^6$$

$$g(a,b) = 6a^5b - 60a^3b^3 + 54ab^5$$

On equating the coefficients of corresponding terms in (21) ,one obtains

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 9| September 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

$$u = 4f(a,b) - 3g(a,b)$$
$$v = f(a,b) + 4g(a,b)$$

In view of (2),we have

$$x = 5f(a,b) + g(a,b)$$
$$y = 3f(a,b) - 7g(a,b) \quad (22)$$

Thus, (22) & (19) satisfy (1).

Process 4

Write (3) as

$$u^2 + 3v^2 = 19\ z^6 * 1 \quad (23)$$

Assume

$$z = 4(a^2 + 3b^2) \quad (24)$$

Consider the integer 1 on the R.H.S. of (23) as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (25)$$

Substitute (20),(24) & (25) in (23) and employ the method of factorization. On equating the positive factors on both sides ,one obtains

$$u + i\sqrt{3}\,v = (4+i\sqrt{3}) * 2^6 [f(a,b) + i\sqrt{3}\,g(a,b)] * \frac{(1+i\sqrt{3})}{2} \quad (26)$$
$$= 2^5 (1+i5\sqrt{3})[f(a,b) + i\sqrt{3}\,g(a,b)]$$

On equating the coefficients of corresponding terms in (26) ,one obtains

$$u = 2^5 [f(a,b) - 15g(a,b)]$$
$$v = 2^5 [5f(a,b) + g(a,b)]$$

In view of (2),we have

$$x = 2^5[6f(a,b) - 14g(a,b)]$$
$$y = 2^5[-4f(a,b) - 16g(a,b)] \quad (27)$$

Thus, (24) & (27) satisfy (1).

Note 1

In addition to (25), the integer 1 has the following representations:

$$1 = \frac{(r^2 - 3s^2 + i\sqrt{3}(2rs))(r^2 - 3s^2 - i\sqrt{3}(2rs))}{(r^2 + 3s^2)^2}$$

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}(2rs))(3r^2 - s^2 - i\sqrt{3}(2rs))}{(3r^2 + s^2)^2}$$

$$1 = \frac{(6s^2 - 6s + 1 + i\sqrt{3}(2s-1))(6s^2 - 6s + 1 - i\sqrt{3}(2s-1))}{(6s^2 - 6s + 2)^2}$$

$$1 = \frac{(2s^2 - 2s - 1 + i\sqrt{3}(2s-1))(2s^2 - 2s - 1 - i\sqrt{3}(2s-1))}{(2s^2 - 2s + 2)^2}$$

Repeating the above process, one obtains four more sets of integer solutions to (1).

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 9| September 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

CONCLUSION

In this paper, an attempt has been made to obtain many integer solutions to non-homogeneous Diophantine equation of degree six with three unknowns. Substitution strategy and factorization technique are utilized for finding the required integer solutions to the sixth degree equation with three unknowns.

REFERENCES

1. *J. Shanthi, M.A. Gopalan, Delineation of Integer Solutions to Non-Homogeneous Quinary Quintic Diophantine Equation* $(x^3-y^3)-(x^2+y^2)+(z^3-w^3)=2+87T^5$, *International Journal of Research Publication and Reviews, 4(9), 1454-1457, 2023.*
2. *J. Shanthi, M.A. Gopalan, On the Ternary Non-homogeneous Quintic Equation* $x^2+5y^2=2z^5$, *International Journal of Research Publication and Reviews, 4(8), 329-332, 2023.*
3. *J. Shanthi, M.A. Gopalan, On the Non-homogeneous Quinary Quintic Equation* $x^4+y^4-(x+y)w^3=14z^2T^3$, *International Research Journal of Education and Technology, 05 (08), 238-245, 2023.*
4. *J. Shanthi , M.A. Gopalan ,Techniques to solve Non-homogeneous Quaternary Sextic Diophantine Equation* $x^3+y^3=7zw^5$, *International Research Journal of Education and Technology, 6(12), 1858-1865, 2024.*
5. *J. Shanthi, N. Thiruniraiselvi, M.A. Gopalan, A Portrayal of integer solutions to Non-Homogeneous ternary cubic Diophantine equation* $6(x^2+y^2)-11xy=2z^3$ *Arhimede Mathematical Journal, Volume 11, No.2, Pp 158-167.Autumn 2024.*
6. *J.Shanthi, M.A.Gopalan, on finding Integer Solutions to Binary Quintic Equation* $x^2-xy^2=y^4+y^5$, *International Journal of Advanced Research in Science, Engineering and Technology, 11(10), 22369-22372, 2024.*
7. *Dr. J. Shanthi, M.A. Gopalan, A Classification of Integer Solutions to Binary Cubic equation* $x^2-xy=3(y^2+y^3)$, *International Journal of Progressive Research in Engineering Management and Science, Volume 05, Issue 05, 1825-1828, May 2025.*
8. *Dr. J. Shanthi, M.A. Gopalan, Observations on Binary Cubic Equation* $x^2-3xy=4(y^2+y^3)$, *International Journal of Advanced Research in Science, Communication and Technology, Volume 5, Issue 1, 1-5, June 2025.*
9. *Dr. J. Shanthi, Dr. N. Thiruniraiselvi, Dr. M.A. Gopalan, Techiques to slove Diophantine Equation of Degree ten with six unknowns* $3(x^2-y^2)^3+4(a^2+b^2)(x^2-y^2)P^2T^6=(z^4-w^4)P^2$, *International Research Journal of Modernization in Engineering Technology and Science, Volume 07, Issue 06, 74-78, June 2025.*
10. *J.Shanthi, M.A.Gopalan, on finding Integer Solutions to Binary Quintic Equation* $x^2-xy^2=y^4+y^5$, *International Journal of Advanced Research in Science, Engineering and Technology, 11(10), 22369-22372, 2024.*
11. *Dr. J. Shanthi, M.A. Gopalan, "On Solving Binary Cubic Equation* $x^2-4xy=5y^3-3y^2$, *International Research journal of education and technology, Volume 08, Issue 06,139-144, June 2025.*
12. *Dr. J. Shanthi, M.A. Gopalan, "On Finding Integer Solutions to Non-homogeneous Ternary Cubic Equation* $(x-1)^2+(y-1)^2=8z^3$, *International Journal of Research Publication and Reviews, Volume 08, Issue 7,312-317, July 2025.*
13. *Dr. J. Shanthi, "On Finding Integer solutions to Binary Cubic equation* $3x^2-xy=y^3$, *Granja Journal, Volume 8, Issue 8, 29-36, August 2025.*
14. *Dr. J. Shanthi, M.A. Gopalan, "A Trek on the Cubic with Four Unknows* $x^2-y^2=z^3-w^3$, *International Research journal of education and technology, Volume 09, 2026.*
15. *N. Thiruniraiselvi, J. Shanthi, M.A. Gopalan, A Trek on the Non- Homogeneous Quaternary Sextic Surface* $x^3+y^3=19zw^5$, *Zhuzao/Foundry, 29 (3), 13-20, 2026*
16. *J. Shanthi, N. Thiruniraiselvi, M.A. Gopalan, Patterns of Integer Solutions to Non- homogeneous Ternary Heptic Diophantine Equation* $x^2+y^2=(a^2+b^2)z^7$, *GongchengKexueXuebao, Volume 11(03),41-47, 2026.*

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 12| Issue: 9| September 2026|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2026: 8.753 || ISI Value: 1.188

17. *N. Thiruniraiselvi, J. Shanthi, M.A. Gopalan, A Glimpse on the Non-homogeneous Quaternary OcticSurface, Gongcheng Kexue Xuebao, Volume 11(03), 88-95, 2026.*
18. *N. Thiruniraiselvi, J. Shanthi, M.A. Gopalan, A Trek on the Non-homogeneous Quaternary Octic Surface* $x^3 + y^3 = 8zw^7$, *Gongcheng Kexue Xuebao, Volume 11(03), 111-117, 2026.*