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PROPERTIES OF τ AND σ FUNCTIONS

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ABSTRACT
In this article, we study τ and σ functions and their properties. Utilizing τ and σ functions, we investigate perfect number, deficient numbers, abundant numbers, amicable numbers, friendly numbers and solitary numbers.

1. INTRODUCTION

τ (tau) and σ (sigma) functions are most important number theoretic functions also known as arithmetic functions. Arithmetic functions are defined for all positive integers. τ and σ functions belong to a large class of arithmetic functions called multiplicative functions and study of some their fascinating properties. τ function, also known as the divisor function, determines the number of positive divisors of a given integer and the σ function also known as the sum of divisors function, calculates the sum of all positive divisors of a given integer. These functions play a significant role in various areas of number theory including the study of prime factorization, divisor properties and properties of perfect numbers. We can use the σ function to study a marvelous class of numbers called perfect numbers the term perfect numbers. The term perfect numbers was coined by Pythagoreans. Euclid shown that every integer $N = 2^{n-1}(2^n - 1)$, where $2^n - 1$ is a prime, is a perfect. Amicable numbers were known to the Pythagoreans, who credited them with many mystical properties.

2. THE FUNCTION τ(N)

2.1 Definition. Let n be a given positive integer. Then $\tau(n)$ is the number of positive divisors of n including the divisors 1 and n.
For an example of these notion, consider n = 12.

The divisors of 12 are 1, 2, 3, 4, 6 and 12.

Hence $\tau(n) = 6$.

For the first few integers,

$\tau(1) = 1, \tau(2) = 2, \tau(3) = 2, \tau(4) = 3, \ldots\ldots$

It is not difficult to see that $\tau(n) = 2$ if and only if n is a prime number.

2.2 Theorem. Let $n = p_1^{a_1}. p_2^{a_2}. p_3^{a_3} \ldots\ldots\ldots p_k^{a_k}$. Then the divisors of n are the various terms of the expansion of

$n = (1 + p_1 + p_1^2 + \ldots\ldots.. + p_1^{a_1})\ (1 + p_2 + \ldots.. + p_2^{a_2}) \ldots\ldots\ldots (1 + p_k + \ldots\ldots + p_k^{a_k})$.

2.3 Theorem. If $n = p_1^{k_1} p_2^{k_2} \ldots\ldots\ldots\ldots P_r^{k_r}$ is the prime factorization of $n > 1$, then

(1) $\tau(n) = (k_1 + 1)(k_2 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (k_n + 1)$ and
(2) $\sigma(n) = \frac{p1^{k1+1}-1}{p1-1}\ \frac{p2^{k2+1}-1}{p2-1} \ldots\ldots\ldots\ldots\ldots \frac{pr^{kr+1}-1}{pr-1}$.

2.4 Example. Let $n = 2700 = 2^2 \times 3^3 \times 5^2$. Then

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$\tau(2700) = (2 + 1)\,(3 + 1)\,(2 + 1) = 36$

$\sigma(2700) = \frac{2^3-1}{2-1} \times \frac{3^4-1}{3-1} \times \frac{5^3-1}{5-1} = 8680$

2.5 Theorem. τ(n) is odd if and only if n is a perfect square.

Proof. Suppose $n = p_1^{a_1}.\ p_2^{a_2}.\ p_3^{a_3} \ldots\ldots\ldots\ p_k^{a_k}$.

(1) Let n be square integer. Then $a_1, a_2, \ldots\ldots\ldots\ a_k$ are all even. Hence

$\tau(n) = (a_1 + 1)\,(a_2 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ (a_k + 1)$
= product of odd factors omly
= an odd integer.

(2) Let τ(n) be odd. Then $(a_1 + 1)\,(a_2 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ (a_k + 1)$ is odd. This implies that $a_1, a_2, \ldots\ a_k$ are all even. It follows that n is a square integer.

2.6 Corollary. Let τ(n) is even if and only if n is not a square integer.

2.7 Definition. A number theoretic function f is said to be multiplicative if

$$f(m\,n) = f(m)\,f(n)$$

whenever, gcd(m, n) = 1.

The function τ and σ are both multiplicative functions.

2.8 Definition. The divisors of an integer n, excluding the divisor n are called proper divisors of n, and the sum of all proper divisors of n is denoted by $\sigma_0(n)$.

Examples

(1) Let n = 12. Then the divisors of 12 are 1, 2, 3, 4, 6, 12. Proper divisors of 12 are 1, 2, 3, 4, 6. Therefore

$\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$

$\sigma_0(12) = 1 + 2 + 3 + 4 + 6 = 16.$

(2) Let n = 28. Then

$\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56$

$\sigma_0(28) = 1 + 2 + 4 + 7 + 14 = 28.$

Here $\sigma(n) = 2n$ and $\sigma_0(n) = n$

2.9 Definition. An integer n is said to be a perfect number if the sum of its proper divisors is equal to n. Thus n is a perfect number of $\sigma_0(n) = n$, or $\sigma(n) = 2n$.

2.10 Example. Let $n = 496 = 2^4 \times 31$. Then

$\sigma(496) = \sigma(2^4)\,\sigma(31) = \frac{2^5-1}{2-1} \times (31 + 1)$

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$= 2 \times 496$

So, 496 is a perfect number.

There is a simple theorem which enables us to find even perfect numbers. The theorem bears the name of Euclid.

2.11 Theorem (Euclid). Let $2^k - 1$ be a prime. Then $2^{k-1}(2^k - 1)$ is a perfect number.

Proof. Let $n = 2^{k-1}(2^k - 1)$. Then, since 2^{k-1} is a prime,

$$\sigma(n) = \sigma(2^{k-1}(2^k - 1) = \frac{2^{k-1}}{2-1} \times (2^k - 1 + 1)$$

$$= (2^k - 1)\, 2^k = 2\left[(2^{k-1}(2^k - 1))\right]$$

Which proves the theorem.

2.12 Definition. n is called a deficient number if $\sigma_0(n) < n$.

Examples

(1) $\sigma(15) = \sigma(5)\,\sigma(3) = 24$. Hence $\sigma_0(15) = 24 - 15 = 9 < 15$.

Therefore 15 is a deficient number.

(2) $\sigma(32) = \sigma(2^5) = 2^6 - 1 = 63$.

Hence $\sigma_0(32) = 63 - 32 = 31 < 32$.

Therefore, 32 is a deficient number.

(3) If p is a prime, there is only one proper devisor of p namely 1. Hence $\sigma_0(p) = 1 < p$. Thus every prime is deficient.

(4) p^k is deficient, for

$$\sigma 0(p^k) = \frac{p^{k-1} - 1}{p - 1} - p^k = \frac{p^k - 1}{p - 1} < p^k$$

(5) If find p and q are primes then pq is deficient.

For, $\sigma 0(pq) = (p+1)(q+1) - pq$

$= p + q + 1$

$= pq - [(p-1)(q-1) - 2]$

$< pq.$

2.13 Definition. n is called an abundant number if $\sigma_0(n) > n$.

Examples. $\sigma 0(12) = 1 + 2 + 3 + 4 + 6 = 16 > 12.$

Hence 12 is an abundant number.

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$$\sigma 0(30) = \sigma(2 \times 3 \times 5) - 30$$

$= (2 + 1)\ (3 + 1)\ (5 + 1) - 30$
$= 42 > 30$
(1) $\sigma(945) = \sigma(3^3 \times 5 \times 7) - 945$
$= 40 \times 6 \times 8 - 945$
$= 975 > 945.$

Hence 945 is an abundant number. It is worth nothing here that 945 is the first odd abundant number.

2.14 Definition. An integer n is called a multiply perfect number of class k if $\sigma_0(n) = kn$.

Example.

I $\sigma 0(120) = \sigma(2^3 \times 3 \times 5) - 120 = 2 \times 120$

Hence 120 is a multiply perfect number of class 2.

II $\sigma 0(2^9 \times 3 \times 11 \times 31) = (2^{10} - 1) \times 4 \times 12 \times 32 - 2^9 \times 3 \times 11 \times 31$

$= 2(2^9 \times 3 \times 11 \times 31)$

Hence $2^9 \times 3 \times 11 \times 31$ is a multiply perfect number of class 2.

2.15 Definition. Let there be two integers a and b such that $\sigma 0(a) = b\ and\ \sigma 0(b) = a$. Then (a, b) is called a pair of amicable numbers.

Example.

I. $\sigma 0(220) = 284\ \ and$
$\sigma 0(284) = 220$

Hence (220, 284) is a pair of amicable numbers. This was the only pair discovered before fermate applied his powerful mind to this subject.

II. It was Euler, however, who undertook a systematic study of amicable number and he gave a list of 60 Such pairs some of these are
   a) $(2 \times 5 \times 7 \times 19 \times 107, \quad 2 \times 5 \times 47 \times 350)$
   b) $(2^3 \times 5 \times 251, \quad 2^2 \times 13 \times 107$
   c) $(2^3 \times 17 \times 79, \quad 2^3 \times 23 \times 59)$

2.16 Theorem. A pair of integers (a, b) is amicable if and only if

$$\sigma(a) = \sigma(b) = a + b$$

Proof.

i. Let $\sigma(a) = \sigma(b) = a + b.$ Then
$\sigma(a) - a = b$ *and* $\sigma(b) - b = a$

This implies that $\sigma 0(a) = b$ and $\sigma 0(b) = a$

Therefore, (a, b) is an amicable pair.

ii. Let (a, b) be an amicable pair. Then
$\sigma 0(a) = b$ and $\sigma 0(b) = a.$ Hence
$\sigma(a) = a + b$ and $\sigma(b) = a + b$

This proves the theorem completely.

2.17 Theorem. Let

$a_k = 2^k(3\times 2^{k-1} - 1)\ (3 \times 2^k - 1)$ and

$b_k = 2^k(9 \times 2^{2k-1} - 1)$

Where $k > 1$. Then (a_k, b_k) is an amicable pair provided the integers.

$$3 \times 2^{k-1} - 1, \quad 3 \times 2^k - 1, \quad 9 \times 2^{2k-1} - 1 \textit{ are primes}.$$

Proof. We have

$a_k + b_k = 2^k(3 \times 2^{k-1} - 1)(3 \times 2^k - 1) + 2^k(9 \times 2^{2k-1} - 1)$

$= 2^k(9 \times 2^{2k-1} - 3 \times 2^k - 3 \times 2^{k-1} + 1 + 9 \times 2^{2k-1} - 1)$

$= 9 \times 2^{3k} - 6 \times 2^{2k-1} - 3 \times 2^{2k-1}$

$= 9 \times 2^{2k-1}(2^{k+1} - 1).$

$\sigma_0(n) = \sigma(2^k)\ \sigma(3 \times 2^{k-1} - 1)\ \sigma(3 \times 2^k - 1)$

Since

$$3 \times 2^{k-1} - 1 \text{ is odd for k1},$$

$$= (2^{k+1} - 1)(3 \times 2^{k-1})(3 \times 2^k)$$

$$= 9 \times 2^{2k-1}(2^{k=1} - 1)$$

$= a_k + b_k$

$\sigma(bk) = \sigma(2^k)\ \sigma(9 \times 2^{2k-1} - 1)$

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$= (2^{k+1} - 1)\ 9 \times 2^{2k-1}$

$= a_k + b_k$

It follows that (a_k, b_k) is an amicable pair.

There exist only three amicable pairs conforming to the above theorem for k ≤ 200. These are

S. No.	k	a_k	b_k
1.	2	220	284
2.	4	17296	18416
3.	7	9363584	9437056

We give below a few more examples of amicable pairs. These are

a) (1184, 1210)
b) (2620, 2924)
c) (5020, 5564)
d) (6232, 6368)

2.18 Abundancy Index. Abundancy index is the ratio of a number's sum of divisors to the number itself.

2.19 Friendly Numbers. Friendly numbers are two or more numbers that have the same abundancy index.

Example

i. 6 and 28 are friendly number.

The devisors of

6 are 1, 2, 3, 6 and

28 are 1, 2, 4, 8, 14, 28.

The abundancy index of 6 is $\frac{\sigma(n)}{n} = \frac{1+2+3+6}{6} = \frac{12}{6}$

$=2$

The abundancy index of 28 is $\frac{\sigma(n)}{n} = \frac{1+2+4+7+14+28}{28} = \frac{56}{28}$ =2

Hence the number 6 and 28 are friendly numbers because they both have an abundancy of 2.

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ii. 30 and 140 are friendly numbers.
iii. 80 and 200 are friendly numbers.

2.20 Solitary Numbers

The numbers that are not part of any friendly pair are called solitary numbers.

Note: Every prime is a solitary number.

Be have our of $\tau(n)$ as n → ∞

I. We know that ever integer $n > 1$ has at least two divisors 1 and n. Hence $\tau(n)$ can never be less than 2 for $n > 1$.
II. Let "p" be a prime. We know that the prime p has only two divisors 1 and p itself. Hence $\tau(p) = 2$.

It follows that $\tau(n)$ assumes the value 2 for infinitely many values of n because the number of prime is infinite.

Thus we write

$$\lim_{n\to\infty} \tau(n) = 2$$

III. If n= p^a, p is a prime, Then $\tau(n) = a + 1$

Therefore $\tau(n)$ assumes values larger than any given number for sufficiently large values of a.

Hence we may write

$$\lim_{n\to\infty} \tau(n) = \infty$$

The function $\tau(n)$ thus behaves in a very irregular way.

Behaviour of σ(n) as he n → ∞

We know that $\sigma(n) > n$ for all $n > 1$. It follows $\sigma(n)$ assumes values larger than any given number for sufficiently large values of n. Thus we have

$$\lim_{n\to\infty} \tau(n) = \infty$$

On the other hand $\sigma(n)$ is not a monotonic increasing functions of n.

2.20 Theorem. Let p be a prime > 3. Then $\sigma(p-1) > \sigma(p)$.

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Proof.

I. Let $p-1=2^t, t>1. Then$
$\sigma(\mathrm{p}-1)=\sigma(2^t)=2^t+2^{t-1}+\cdots+2+1$

Hence

$\sigma(p-1)> \ 2^t+2>p+1= \ \sigma(p).$

II. Let $(p-1)=p1^{a1}p2^{a2} \ldots.. \ \ pk^{ak}. \qquad Then$

$\sigma(p-1)=\sigma(p1^{a1})\,\sigma(p2^{a2}) \ldots \ldots \ldots \ldots \ldots.\sigma(pk^{ak})$

$\geq (p1^{a1}+1)(p2^{a2}+1) \ldots \ldots \ldots \ldots..(p1^{a1}+1)$

$\geq p1^{a1}p2^{a2} \ldots \ldots \ldots \ldots pk^{ak}+(p1^{a1}+p2^{a2} \ldots \ldots..+pk^{ak})$

$\geq p-1+k+1$

$> p+1 \qquad \qquad Since\ k>1$

$= \sigma(p)$

Thus $\sigma(n)$ is an irregular function of n although not as $\tau(n)$

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