ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 11| Issue: 6| June 2025|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2025: 8.691 || ISI Value: 1.188

ON THE TERNARY SURD EQUATION $x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z}$

Dr.J.Shanthi[1*], Dr.N.Thiruniraiselvi[2], Dr. M.A.Gopalan[3]

[1,2,3] Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

Orcid id[1]: 0009-0008-5945-410X ;Orcid id[2]: 0000-0003-4652-3846 ; Orcid id[3]: 0000-0003-1307-2348*

Article DOI: https://doi.org/10.36713/epra22213
DOI No: 10.36713/epra22213

ABSTRACT

The transcendental equation with three unknowns given by $x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z}$ *is studied for obtaining its non-zero integer solutions.*

INTRODUCTION

The subject of Diophantine equation, one of the interesting areas of Number Theory, plays a significant role in higher arithmetic and has a marvellous effect on credulous people and always occupies a remarkable position due to unquestioned historical importance. The Diophantine equations may be either polynomial equation with at least two unknowns for which integer solution, are required or transcendental equation involving trigonometric, logarithmic, exponential and surd function such that one may be interested in getting integer solution. Most of the Diophantine problems solved by the researchers are polynomial equations [3-7]. Note that, the transcendental equation can be solved by transforming it into an equivalent polynomial equation. Adhoc methods exists for some classes of transcendental equations in one variable to transform them into polynomial equations which then might be solved. Some transcendental equation in more than one unknown can be solved by separation of the unknowns reducing them to polynomial equations. In this context, one may refer [1,2,8-15].

In this paper, we are interested in obtaining integer solutions to transcendental equation involving surds. In particular, we obtain different sets of integer solutions to the transcendental equation with three unknowns given by $x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z}$.

Method of analysis

The surd equation with three unknowns under consideration is

$$x+\sqrt{x}+2(y+\sqrt{y})=z+\sqrt{z} \tag{1}$$

The substitution

$$x=u^2, y=v^2, z=w^2 \tag{2}$$

in (1) leads to the ternary quadratic equation

$$(2u+1)^2+2(2v+1)^2=(2w+1)^2+2 \tag{3}$$

Two different methods of solving (3) are illustrated below:

Method 1

Let

$$2u+1=U,\ 2v+1=V,\ 2w+1=P. \tag{4}$$

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 11| Issue: 6| June 2025|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2025: 8.691 || ISI Value: 1.188

On completing the squares, (3) is written as

$$U^2 + 2V^2 = (P^2 + 2) * 1 \tag{5}$$

Assume

$$1 = \frac{\left(2r^2 - s^2 + i2rs\sqrt{2}\right)\left(2r^2 - s^2 - i2rs\sqrt{2}\right)}{\left(2r^2 + s^2\right)^2} \tag{6}$$

Substituting (6) in (5) and applying factorization, consider

$$U + i\sqrt{2}V = (P + i\sqrt{2}) * \frac{(2r^2 - s^2 + i2\sqrt{2}rs)}{(2r^2 + s^2)}$$

Equating the coefficients of corresponding terms and using (4) for P, we have

$$\begin{aligned} U &= \frac{1}{(2r^2 + s^2)}[2(2r^2 - s^2)w + 2r^2 - s^2 - 4rs] \\ V &= \frac{1}{(2r^2 + s^2)}[2r^2 - s^2 + 2rs + 4rsw] \end{aligned} \tag{7}$$

It is possible to choose the values for r, s, w, such that U, V are integers. In view of (4) and (2), the corresponding integer solutions to (1) are given by

r	s	w	u	v	x	y	z
1	1	3s+3	s	2s+2	s^2	$(2s+2)^2$	$(3s+3)^2$
2	3	17s+13	-s-2	12s+9	$(-s-2)^2$	$(12s+9)^2$	$(17s+13)^2$

Method 2

Taking

$$U = 2u + 1, V = 2v + 1, P = 2w + 1 \tag{8}$$

in (3), we have

$$U^2 + 2V^2 = P^2 + 2 \tag{9}$$

Choosing

$$P = k\,V, k > 1 \tag{10}$$

in (9), we get

$$U^2 = (k^2 - 2)\,V^2 + 2 \tag{11}$$

which is satisfied by

$$V_0 = 1,\ U_0 = k$$

ISSN (Online): 2455-3662
EPRA International Journal of Multidisciplinary Research (IJMR) - Peer Reviewed Journal
Volume: 11| Issue: 6| June 2025|| Journal DOI: 10.36713/epra2013 || SJIF Impact Factor 2025: 8.691 || ISI Value: 1.188

To obtain the other solutions, consider the Pell equation,

$$U^2 = (k^2 - 2)\, V^2 + 1$$

whose initial solutions are

$$\widetilde{V}_0 = k,\ \widetilde{U}_0 = k^2 - 1$$

and, the general solution (U_n, V_n) is given by

$$U_{n+1} = \frac{k f_n}{2} + \frac{\sqrt{k^2-2}}{2} g_n\, , V_{n+1} = \frac{1}{2} f_n + \frac{k g_n}{2\sqrt{k^2-2}} \tag{12}$$

where

$$f_n = (k^2 - 1 + k\sqrt{k^2-1})^{n+1} + (k^2 - 1 - k\sqrt{k^2-1})^{n+1},$$
$$g_n = (k^2 - 1 + k\sqrt{k^2-1})^{n+1} - (k^2 - 1 - k\sqrt{k^2-1})^{n+1}$$

In view of (8), it is seen that

$$u_{n+1} = \frac{\left(k f_n + \left(\sqrt{k^2-2}\right) g_n - 2\right)}{4},$$

$$v_{n+1} = \frac{\left(f_n + \frac{k}{\left(\sqrt{k^2-2}\right)} g_n - 2\right)}{4},$$

$$w_{n+1} = \frac{\left(k f_n + \frac{k^2}{\left(\sqrt{k^2-2}\right)} g_n - 2\right)}{4},\ n = 0,1,2,...$$

Using the above values in (2), the integer solutions are obtained when k is odd.

CONCLUSION

In this paper, the transcendental equation involving surds with three unknowns given by $x + \sqrt{x} + 2(y + \sqrt{y}) = z + \sqrt{z}$ has been reduced to polynomial equations of degree two with three unknowns as well as pellian equation for which integer solutions are obtained in an elegant way through suitable transformations.

REFERENCES

[1]. M.A.Gopalan, S.Vidhyalakshmi and J.Shanthi, "On the Surd Equation $A\sqrt[a]{x} + B\sqrt[b]{y} = C\sqrt[c]{z}, (a,b,c \in Q)$", International Journal of Development Research, Vol. 06, Issue 10, , 9665-9668 , October 2016. https://journalijdr.com/sites/default/files/issue-pdf/6520.pdf

[2]. M.A.Gopalan, S.Vidhyalakshmi and J.Shanthi, "On the Transcendental Equation with Six Unknowns $\sqrt[2]{x^2+3y^2}+\sqrt[4]{X^3+Y^3}=z^2+w^2$", International Journal of Innovative Research in Science, Engineering and Technology, Vol. 5, Issue 8,14385-14388, August 2016.
https://www.ijirset.com/upload/2016/august/11_ON.pdf

[3] Thiruniraiselvi N, Gopalan M. A. "On the positive pell equation $y^2=11x^2+22$". Academic journal of applied mathematical science.2020; 6(7):85-92.
Available from: DOI: doi.org/10.32861/ajams.67.85.92

[4] Vijayasankar A, Sharadha Kumar, Gopalan M.A. "On the Ternary Quadratic Equation $x^2+y^2=z^2+141$". Academic journal of applied mathematical science.2020; 6(7):80-84. Available from: DOI: doi.org/10.32861/ajams.67.80.84

[5] Thiruniraiselvi N, Gopalan M. A, Poovarasan E. "On the non-homogenous quadratic with two unknowns $5x^2-2y^2=117$". International Journal of Advanced Multidisciplinary Research and Studies.2022; 2(3):270-273.
Available from:https://www.multiresearchjournal.com/arclist/list-2022.2.3/id-244

[6] Janaki G, Gowri Shankari A. "Integral Solution of the Binary Quadratic Equation $3x^2-y^2=2$ that Depicts Parabolas and Hyperbolas". International Journal of Scientific Research in Mathematical and Statistical Sciences. 2023; 10(01):10-13.
Available from:https://www.isroset.org/journal/IJSRMSS/archive_issue.php?pub_id=408

[7] Thiruniraiselvi N, Gopalan M. A. "A class of new solutions in integers to Ternary Quadratic Diophantine Equation $12(x^2+y^2)-23xy+2x+2y+4=56z^2$". International Journal of Research Publication and Reviews. 2024; 5(5):3224-3226.
Available from: https://ijrpr.com/uploads/V5ISSUE5/IJRPR27526.pdf

[8] Meena K, Gopalan M.A, Srilekha J. "On the Transcendental equation with three unknowns $2(x+y)-3\sqrt{xy}=(k^2+7s^2)z^2$". International Journal of engineering Science and Research Technology. 2019; 8(1):133-136.
Available from: https://www.ijesrt.com/Old_IJESRT/January-2019.html

[9] Vidhyalakshmi S, Mahalakshmi T, Gopalan M.A, "On the Transcendental equation $\sqrt[3]{x^2+y^2}+\sqrt[2]{mx+ny}=10z^3$". International Journal of Recent Engineering Research and Development (IJRERD).2020; 5(6): 8-11.
Available from: http://www.ijrerd.com/volume5-issue6.html

[10] Vidhyalakshmi S, Mahalakshmi T, Gopalan M.A. "On the Transcendental Equation $\sqrt{y^2+2x^2}+\sqrt[3]{X^2+Y^2}=35z^3$".International Research Journal of Modernization in Engineering Technology and Science. 2020; 2(8):824-827.
Available from: 1628083113.pdf (irjmets.com)

[11] Vidhyalakshmi S, Gopalan M.A. "On the Transcendental equation $\sqrt[3]{x^2+y^2}=z^6$". International Journal of Advanced Multidisciplinary Research and Studies.2021; 1(3):41-44.
Available from:
https://www.multiresearchjournal.com/arclist/list-2021.1.3/id-69

[12] Vidhyalakshmi S, Gopalan M.A. "Observations on the Surd Equation $\sqrt{2z-4}=\sqrt{x+\sqrt{m^2+4}\,y}+\sqrt{x-\sqrt{m^2+4}\,y}$". International Journal of Research Publication and Reviews.2021; 2(11):437-441.
Available from: Observations on the Surd Equation (ijrpr.com)

[13] Meena K, Vidhyalakshmi S,Gopalan M.A. "On The Surd Equation $\sqrt{2z}=\sqrt{x+ay}+\sqrt{x-ay}\,(a\neq 0)$". International Journal of Modern Developments in Engineering and Science.2022; 1(1):6-7.

Available from: View of On the Surd Equation (ijmdes.com)

[14]. Saranya P, Shri Padmapriya M. "Solutions of the Transcendental Equation $p^2 - \sqrt[3]{p^5 + q^5 - (pq)^3} + \sqrt[3]{r^2 + s^2} = k^3(n^2 + 1)$ *". International Journal for Research in Applied Science & Engineering Technology (IJRASET). 2023; 11(III): 599-603.*
Available from DOI Link: https://doi.org/10.22214/ijraset.2023.49478

[15]. N.Thiruniraiselvi, M.A.Gopalan, "On Finding Integer Solutions to Transcendental Equation with Six Unknowns $\sqrt[3]{X + Y} + \sqrt{xy} = z^2 + w^2$ *", Indian Journal of Science and Technology, Vol. 17(19), 2992-3001, 2024.*
Available from: https://indjst.org/articles/a-portrayal-of-integer-solutions-to-transcendental-equation-with-six-unknowns-