THE ROLE OF CONTINUED FRACTIONS IN SOLVING PELL-TYPE EQUATIONS VIA (γ,δ)-JACOBSTHAL AND(γ,δ) - JACOBSTHAL -LUCAS SEQUENCES

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ABSTRACT

In this manuscript, the techniques for finding the solutions in terms of $(\gamma,\delta)-$ *Jacobsthal and* $(\gamma,\delta)-$ *Jacobsthal Lucas sequences for two distinct categories of the Diophantine equations* $u_n^2-(p^2\pm q)v_n^2=q^{2n}$ *and* $u_n^2-(4p^2q^2-8\ )v_n^2=(-2\delta)^n$*,* $n\geq 1$*,* $\delta\in Z-\{0\}$ *where p and q are specialised prime numbers, including Gaussian prime, Good prime, Lucky prime, Chen prime and Circular prime are offered. Moreover, the entire solutions discovered are confirmed with the assistance of three distinct Python codes.*

KEYWORDS: $(\gamma,\delta)-$ *Jacobsthal,* $(\gamma,\delta)-$ *Jacobsthal Lucas, Gaussian prime, Good prime, Lucky prime, Chen prime, Circular prime.*

1. INTRODUCTION

There is plenty of research about the Jacobsthal and Jacobsthal-Lucas sequences, as well as others, like the Fibonacci sequence, Pell sequence, Pell-Lucas sequence, and their generalised sequences. The preliminary theorems in [1] stated in the following section are required for the main theorems. In [3], Guney employed the generalised Fibonacci sequence and Lucas sequence to determine non-negative integer solutions of $x^2-(a^2b^2+2b)y^2=N,\ N=\pm1,\pm4$. In [5], the Author found the solutions of the Pell equation $x^2-(a^2+2a)y^2=N$ through generalized Fibonacci and Lucas numbers. In [7], the authors established some features of the Jacobsthal and Jacobsthal Lucas sequences, as well as imperative relations among the $(s,t)-$Jacobsthal and $(s,t)-$Jacobsthal Lucas sequences. In [8], Jafari-Petroudia, S.H. Pirouzb acquired the formula for the n^{th} term, the sum of the first n^{th} terms, and the properties of the $(k,h)-$Pell sequence and $(k,h)-$Pell-Lucas. For deeper information regarding these sequences, see [2,4,6,9 − 13].

In this endeavour, the solutions of the Diophantine equations

$u_n^2-(p^2\pm q)v_n^2=q^{2n}$ and $u_n^2-(4p^2q^2-8\ )v_n^2=(-2\delta)^n$, $n\geq 1,\ \delta\in Z-\{0\}$ in terms of $(\gamma,\delta)-$ Jacobsthal and $(\gamma,\delta)-$ Jacobsthal Lucas are scrutinized. Furthermore, the solutions are authenticated by different Python codes.

2. PREPARATORY DEFINITIONS AND THEOREMS

This part presents essential facts that underpin the main theorems.

“Definition 2.1

A positive integer is said to be a Gaussian prime if it satisfies the following conditions:

(i) It is a prime number.
(ii) It is congruent to 3 modulo 4, i.e., it can be expressed in the form $4n+3$, where n is a nonnegative integer.

Samples of Gaussian numbers are 3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83, ...

Definition 2.2

A good prime is a prime p such that p^2 exceeds the product of its symmetric neighbours in the prime sequence.
Examples of a list of some Good primes are 5,11,17,29,37,41 ...

Definition 2.3

A prime number appears in the sequence of Lucky numbers, generated by iteratively removing numbers from the set of natural numbers based on their position.
The process is illustrated as follows.

1. Starting with a list of integers: 1, 2, 3, 4, 5, 6,7,8,9,10,11,12,13,14,15,16,17, . ..
2. Removing every second number, the remaining sequence is 1,3,5,7,9,11,13,15,17, ...

3. Take out every third number, and the sequence becomes $1,3,7,9,13,15,19,21,\ 25,27,31,\ldots$
4. Omit every seventh number, 1,3,7,13,15,19,25,27,31 ...
5. Continuing to remove every n^{th} remaining number, resulting in a sequence of Lucky numbers as $1,3,7,9,13,15,21,25,31,33,37,\ldots$

A prime number appearing in the above sequence of Lucky numbers, such as $3,7,13,31,37,43,\ldots$ It is called a Lucky prime sequence.

Definition 2.4

A circular prime is a prime number that remains prime under all rotations of its digits.

Example: 113 is a circular prime since all its rotations, such as 113, 131, and 311, are prime.

A circular prime is a prime number that remains prime when

1. Its digits are rotated one position to the right (or left)
2. The resulting number is still prime

Example: 1193 is a circular prime because

- 1193 is prime
- Rotating its digits to the right gives 1931, which is prime.
- Rotating again gives 9311, a prime.
- Rotating again gives 3119, which is prime.

Definition 2.5

Chen prime is a prime number p such that $p\ +\ 2$ is either a prime number or can be expressed as the product of two prime numbers.

The first few Chen primes are $2,3,5,7,11,13,17,\ldots$

Theorem 2.1

If $l = 2a$, for all $a > 0$ be the length of the period, then $u^2 - \mathcal{D}v^2 = -1$ has no positive integer solution and the fundamental solution of $u^2 - \mathcal{D}v^2 = K$ is $\frac{p_{n-1}}{q_{n-1}}$.

Theorem 2.2

If $\mathcal{D} > 0$ which is a square-free number, the Diophantine equation $u^2 - \mathcal{D}v^2 = 1$ has infinitely many solutions. Then, all non-negative integer solutions of u and v are obtained by $u_n + \sqrt{\mathcal{D}}\, v_n = \left(u_1 + \sqrt{\mathcal{D}}\, v_1\right)^n, n > 1$, where (u_1, v_1) is the fundamental solution of the equation $u^2 - \mathcal{D}v^2 = 1$."

3. EVALUATION OF INTEGER SOLUTIONS TO PELL-TYPE EQUATIONS BY EMPLOYING A CONTINUED FRACTION

The (γ,δ)-Jacobsthal sequence is categorised by $J_n(\gamma,\delta)$ and its recurrence relation is defined by

$$J_{n+2}(\gamma,\delta) = \gamma J_{n+1}(\gamma,\delta) + 2\delta J_n(\gamma,\delta), \quad J_0(\gamma,\delta) = 0,\ J_1(\gamma,\delta) = 1.$$

The (γ,δ)-Jacobsthal-Lucas sequence is specified by $j_n(\gamma,\delta)$ and its recurrence relation is expressed as

$$j_{n+2}(\gamma,\delta) = \gamma j_{n+1}(\gamma,\delta) + 2\delta j_n(\gamma,\delta), \text{ with the initial conditions}$$

$$j_0(\gamma,\delta) = 2,\ j_1(\gamma,\delta) = \gamma \text{ with } n \geq 0$$

The general formula for these two sequences is defined by $J_n(\gamma,\delta) = \frac{\lambda^n - \mu^n}{\lambda-\mu}$ and

$j_n(\gamma,\delta) = \lambda^n + \mu^n$ where $\lambda = \frac{\gamma+\sqrt{\gamma^2+8\delta}}{2}$, $\mu = \frac{\gamma-\sqrt{\gamma^2+8\delta}}{2}$ are the root of the equation $x^2 - \gamma x - 2\delta = 0$ such that $\lambda + \mu = \gamma$, $\lambda - \mu = \sqrt{\gamma^2 + 8\delta}$ and $\lambda\,\mu = -2\delta$.

In this section, the (γ,δ)-Jacobsthal and the (γ,δ)-Jacobsthal-Lucas sequences are employed to analyse the solutions to the Diophantine equation $u_n^2 - \mathcal{D}v_n^2 = q^{2n}$, $n \geq 1$under two cases, namely $\mathcal{D} = p^2 + q$ with p, q are Gaussian prime and a Good prime, respectively and $\mathcal{D} = p^2 - q$ with p, q are Lucky prime and Gaussian prime, respectively. Also, the solutions in terms of the above-mentioned sequences to the Diophantine equation $u_n^2 - (4p^2q^2 - 8\ )v_n^2 = (-2\delta)^n$, $n \geq 1$, $\delta \in Z - \{0\}$ with p, q are Chen prime and Circular prime respectively is scrutinized.

Theorem 3.1

If $\mathcal{D} = p^2 + q$ where p is Gaussian prime and q is a Good prime such that $\mathcal{D}$ is not a perfect square, then

(i) $\sqrt{\mathcal{D}} = \left[p; \overline{\frac{2p}{q}, 2q}\right]$

(ii) The fundamental solution of $u_n^2 - \mathcal{D}v_n^2 = q^{2n}$ is $(u_1, v_1) = (2p^2 + q, 2p)$

(iii) All positive integer solutions of $u_n^2 - (p^2 + q)v_n^2 = q^{2n}$ are given by

$$(u_n, v_n) = \left(\frac{j_n(\gamma,\delta)}{2},\ 2pJ_n(\gamma,\delta)\right) \text{ where } \gamma = 4p^2 + 2q \text{ and } \delta = -\frac{q^2}{2}$$

Proof

(i)
$$\sqrt{\mathcal{D}} = p + \sqrt{(p^2+q)} - p$$
$$= p + \cfrac{1}{\frac{2p}{q} + \frac{\left(\sqrt{p^2+q} - p\right)}{q}}$$
$$= p + \cfrac{1}{\frac{2p}{q} + \cfrac{1}{\left(2p + \sqrt{p^2+q} - p\right)}}$$

(ii) It is determined that the length of the period is 2. Then, the relation $\frac{p_1}{q_1} = p + \frac{1}{\frac{2p}{q}}$ produce the lowest probable solution of $u_n^2 - (p^2+q)v_n^2 = q^{2n}$ is given by

$\left(u_{1,}v_1\right) = (2p^2 + q, 2p)$.

(iii) All the integer solutions of $u_n^2 - (p^2+q)v_n^2 = q^{2n}$ provided by

$$u_n + \sqrt{\mathcal{D}}v_n = \left(2p^2 + q + 2p\sqrt{\mathcal{D}}\right)^n \text{ and } u_n - \sqrt{\mathcal{D}}v_n = \left(2p^2 + q - 2p\sqrt{\mathcal{D}}\right)^n$$

Let us assume that the transformation is

$$\lambda = 2p^2 + q + 2p\sqrt{\mathcal{D}} \text{ and } \mu = 2p^2 + q - 2p\sqrt{\mathcal{D}}.$$

The general solutions to the equation $u_n^2 - (p^2+q)v_n^2 = q^{2n}$are then expressed in terms of $(\gamma, \delta) -$ Jacobsthal and $(\gamma, \delta) -$ Jacobsthal Lucas sequences, which incorporate the Gaussian and Good primes, are described below.

$$u_n = \frac{\lambda^n + \mu^n}{2} = \frac{1}{2}j_n\left(4p^2 + 2p, -\frac{q^2}{2}\right)$$

$$v_n = \frac{\lambda^n - \mu^n}{2\sqrt{\mathcal{D}}} = 2pJ_n\left(4p^2 + 2p, -\frac{q^2}{2}\right)$$

The sample solutions for the given theorem are highlighted in Table 1.

Table 1

p	q	n	$\mathscr{j}_n$	J_n	u_n	v_n	$u_n^2 - \mathcal{D}v_n^2$	q^{2n}
3	5	1	46	1	23	6	25	25
		2	2066	46	1033	276	625	625
		3	93886	2091	46943	12546	15625	15625
	11	1	58	1	29	6	121	121
		2	3122	58	1561	348	14641	14641
		3	174058	3243	87029	19458	1771561	1771561
7	5	1	206	1	103	14	25	25
		2	423886	206	21193	2884	625	625
		3	8726366	42411	4363183	593754	15625	15625
	11	1	218	1	109	14	121	121
		2	47282	218	23641	3052	14641	14641
		3	10281098	47403	5140549	663642	1771561	1771561

A practical illustration of the theorem 3. 1, aligned with our hypothesis, is provided by the succeeding Python code 1.

Python code 1:

```
from decimal import Decimal, getcontext

getcontext().prec = 40

def jacobsthalLucas(n, s, t):

  if n < 0:

    return "Incorrect input"

  elif n == 0:

    return 2

  elif n == 1:

    return s

  else:

    a, b = 2, s

    for _ in range(2, n + 1):

      a, b = b, s * b + 2 * t * a

    return b

def Jacobsthal(n, s, t):
```

```
  if n < 0:
    return "Incorrect input"
  elif n == 0:
    return 0
  elif n == 1:
    return 1
  else:
    a, b = 0, 1
    for _ in range(2, n + 1):
      a, b = b, s * b + 2 * t * a
    return b
def calculate_values(p, q, n):
  D = p ** 2 + q
  s = 4 * p ** 2 + 2 * q
  t = −q ** 2 / 2
  jacobsthalLucas_n = jacobsthalLucas(n, s, t)
  Jacobsthal_n = Jacobsthal(n, s, t)
  u_n = jacobsthalLucas_n / 2
  u2 = u_n ** 2
  v_n = 2 * p * Jacobsthal_n
  v2 = v_n ** 2
  diff = u2 − D * v2
  q_power = q ** (2 * n)
  return jacobsthalLucas_n, Jacobsthal_n, u_n, v_n, diff, q_power
p = int(input("Enter the value of p: "))
q = int(input("Enter the value of q: "))
n = int(input("Enter the value of n: "))
jacobsthalLucas_n, Jacobsthal_n, u_n, v_n, diff, q_power = calculate_values(p, q, n)
print(f"jacobsthalLucas(n) = {jacobsthalLucas_n}")
print(f"Jacobsthal(n) = {Jacobsthal_n}")
```

```
print(f"u(n) = {u_n}")
print(f"v(n) = {v_n}")
print(f"u(n)^2 - D * v(n)^2 = {diff}")
print(f"q^(2n) = {q_power}")
if diff == q_power:
  print("u and v are the solutions of the given equation.")
else:
  print("u and v are not the solutions of the given equation.")
```

Theorem 3.2

If $\mathcal{D} = p^2 - q$ is not a perfect square where p is a Lucky prime, q is a Gaussian prime, then

i) $\sqrt{\mathcal{D}} = \left[p; \overline{\frac{-2p}{q}, 2q}\right]$

ii) The basic solution of $u_n^2 - \mathcal{D}v_n^2 = q^{2n}$ is $(u_1, v_1) = (2p^2 - q, 2p)$

iii) All positive integer solutions of $u_n^2 - (p^2 - q)v_n^2 = q^{2n}$ are given by (u_n, v_n) for $n \geq 1$ with $u_n = \frac{j_n(\gamma,\delta)}{2}$ and $v_n = 2pJ_n(\gamma, \delta)$ where $\gamma = 4p^2 - 2q$ and $\delta = -\frac{q^2}{2}$

Proof

The proof of Theorem 3.2 is quite similar to that of the theorem 3.1.

The subsequent table 2 offers illustrative solutions for the stated theorem 3.2.

Table 2

p	q	n	j_n	J_n	u_n	v_n	$u_n^2 - Dv_n^2$	q^{2n}
3	3	1	30	1	15	6	9	9
		2	882	30	441	180	81	81
		3	26190	891	13095	5346	729	729
	11	1	14	1	7	6	121	121
		2	−46	14	−23	84	14641	14641
		3	−2338	75	−1169	450	1771561	1771561
13	7	1	662	1	331	26	49	49
		2	438146	662	219073	17212	2401	2401
		3	290020214	438195	145010107	11393070	117649	117649
	19	1	638	1	319	26	361	361
		2	406322	638	203161	16588	130321	130321
		3	259003118	406683	129501559	10573758	47045881	47045881

The following is a compact representation of Python code 2, designed to validate our stated condition in theorem 3.2.

Python code 2

```
from decimal import Decimal, getcontext

getcontext().prec = 50

def Jacobsthal(n, s, t):

  if n < 0:

    raise ValueError("n must be non − negative")

  elif n == 0:

    return Decimal(0)

  elif n == 1:

    return Decimal(1)

  else:

    a, b = Decimal(0), Decimal(1)

    for _ in range(2, n + 1):

      a, b = b, s * b + 2 * t * a

    return b

def JacobsthalLucas(n, s, t):

  if n < 0:

    raise ValueError("n must be non − negative")

  elif n == 0:

    return Decimal(2)

  elif n == 1:

    return Decimal(s)

  else:

    a, b = Decimal(2), Decimal(s)

    for _ in range(2, n + 1):

      a, b = b, s * b + 2 * t * a

    return b

def calculate_values(p, q, n):

  D = Decimal(p ** 2 − q)
```

```
    s = Decimal(4 * p ** 2 - 2 * q)
    t = Decimal(-q ** 2) / 2
    Jacobsthal_n = Jacobsthal(n, s, t)
    JacobsthalLucas_n = JacobsthalLucas(n, s, t)
    u_n = JacobsthalLucas_n / 2
    u2 = u_n ** 2
    v_n = 2 * p * Jacobsthal_n
    v2 = v_n ** 2
    diff = u2 - D * v2
    q_power = Decimal(q) ** (2 * n)
    return D, s, t, Jacobsthal_n, JacobsthalLucas_n, u_n, v_n, diff, q_power
def display_results(p, q, n):
    D, s, t, Jacobsthal_n, JacobsthalLucas_n, u_n, v_n, diff, q_power = calculate_values(p, q, n)
    print(f"\nFor n = {n}: ")
    print(f"Jacobsthal(n) = {Jacobsthal_n}")
    print(f"JacobsthalLucas(n) = {JacobsthalLucas_n}")
    print(f"u(n) = {u_n}")
    print(f"v(n) = {v_n}")
    print(f"u(n)^2 - D * v(n)^2 = {diff}")
    print(f"q^(2n) = {q_power}")
    if diff == q_power:
        print("u and v are the solutions of the given equation.")
    else:
        print("u and v are not the solutions of the given equation.")
try:
    p = int(input("Enter the value of p: "))
    q = int(input("Enter the value of q: "))
    default_ns = [1, 2, 3]
    for n in default_ns:
        display_results(p, q, n)
```

```
  custom_n = input("Enter a custom value for n (or press Enter to skip): ")
  if custom_n.strip():
    n = int(custom_n)
    display_results(p, q, n)
except ValueError as e:
  print(f"Error: {e}")
```

Theorem 3.3

If $\mathcal{D} = 4p^2q^2 - 8$ where p is a Chin prime and q is a Circular prime such that $\mathcal{D}$ is a square-free number, then

i) $\sqrt{\mathcal{D}} = \left[2pq-1; \overline{\frac{4pq-2}{4pq-9}, 2(2pq-1)}\right]$

ii) The least positive integer solution of $u_n^2 - \mathcal{D}v_n^2 = (-2\delta)^{\boldsymbol{n}}$ is
$(u_1, v_1) = (8p^2q^2 - 4pq - 7, 4pq - 2)$

iii) All other positive integer solutions of $u_n^2 - \mathcal{D}v_n^2 = (-2\delta)^{\boldsymbol{n}}$ are specified by (u_n, v_n) for $n \geq 1$ with
$u_n = \frac{1}{2}\jmath_n\left(16p^2q^2 - 8pq - 14,\ \frac{1}{2}(-16p^2q^2 + 72pq - 81)\right)$ and
$v_n = (4pq-2)J_n\left(16p^2q^2 - 8pq - 14,\ \frac{1}{2}(-16p^2q^2 + 72pq - 81)\right)$

Proof

The proof provided for Theorem 3.1 is homogeneous to that of Theorem 3.3 and the numerical evidence of a specific theorem 3.3 is offered in the Table 3.

Table 3

p	q	n	$\jmath_n$	J_n	u_n	v_n	$u_n^2 - \mathcal{D}v_n^2$	$(-2\delta)^n$
2	2	1	210	1	105	14	49	49
		2	44002	210	22001	2940	2401	49^2
		3	9230130	44051	4615065	616714	117649	49^3
	3	1	514	1	257	22	225	225
		2	263746	514	131873	11308	50625	225^2
		3	135449794	263971	67724897	5807362	1190625	225^3
5	2	1	1506	1	753	38	961	961
		2	2266114	1506	1133057	57228	923521	961^2
		3	3411320418	2267075	1705660209	86148850	887503681	961^3
	3	1	3466	1	1733	58	2601	2601
		2	12007954	3466	6003977	201028	6765201	2601^2
		3	41610553498	12010555	20805276749	696612190	17596287801	2601^3

The Python code 3 below provides a concrete implementation of our proposed solutions in theorem 3.3.

Python code 3:

```
from decimal import Decimal, getcontext
def jacobsthalLucas(n, s, t):
  if n < 0:
    return "Incorrect input"
  elif n == 0:
    return 2
  elif n == 1:
    return s
  else:
    a, b = 2, s
    for _ in range(2, n + 1):
      a, b = b, s * b + 2 * t * a
    return b
def Jacobsthal(n, s, t):
  if n < 0:
    return "Incorrect input"
  elif n == 0:
    return 0
  elif n == 1:
    return 1
  else:
    a, b = 0, 1
    for _ in range(2, n + 1):
      a, b = b, s * b + 2 * t * a
    return b
def calculate_values(p, q, n):
  getcontext().prec = 40 + n
  D = Decimal(4 * p ** 2 * q ** 2 − 8)
  S = Decimal(16 * p ** 2 * q ** 2 − 8 * p * q − 14)
  two_t = Decimal(−16 * p ** 2 * q ** 2 + 72 * p * q − 81)
```

```
    t = two_t / 2 # Calculate t from 2t
    jacobsthalLucas_n = Decimal(jacobsthalLucas(n, S, t))
    Jacobsthal_n = Decimal(Jacobsthal(n, S, t))
    u_n = jacobsthalLucas_n / 2
    u2 = u_n ** 2
    v_n = Decimal(4 * p * q - 2) * Jacobsthal_n
    v2 = v_n ** 2
    diff = u2 - D * v2
    two_t_power = (-2 * t) ** n
    return D, S, two_t, t, jacobsthalLucas_n, Jacobsthal_n, u_n, v_n, diff, two_t_power
p = int(input("Enter the value of p: "))
q = int(input("Enter the value of q: "))
n = int(input("Enter the value of n: "))
D, S, two_t, t, jacobsthalLucas_n, Jacobsthal_n, u_n, v_n, diff, two_t_power = calculate_values(p, q, n)
print(f"jacobsthalLucas(n) = {jacobsthalLucas_n}")
print(f"Jacobsthal(n) = {Jacobsthal_n}")
print(f"u(n) = {u_n}")
print(f"v(n) = {v_n}")
print(f"u(n)^2 - D * v(n)^2 = {diff}")
print(f"(-2t)^n = {two_t_power}")
print(f"Step - by - step verification for n = {n}: ")
print(f"LHS (u_n^2 - D * v_n^2) = {diff}")
print(f"RHS (-2t)^n = {two_t_power}")
print(f"Difference (LHS - RHS) = {diff - two_t_power}")
if diff == two_t_power:
    print("\nVerification successful: u and v are the solutions of the given eqn.")
else:
    print("\nVerification failed: u and v are not the solutions of the given eqn.")
```

4. CONCLUSION

This study addresses the solutions to the Pell-type equations $u_n^2 - \mathcal{D}v_n^2 = q^{2n}$ and $u_n^2 - \mathcal{D}v_n^2 = (-2\delta)^n$ with a continued fraction expansion of $\sqrt{\mathcal{D}}$ using $(\gamma, \delta)-$ Jacobsthal and $(\gamma, \delta)-$ Jacobsthal-Lucas sequences. All such solutions are verified by Python codes. Furthermore, the integer solutions to the Diophantine problems can be found by investigating different combinations for D.

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