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CONTINUATION OF HARMONIC FUNCTIONS WITH DELICATE SPECIAL PROPERTIES

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ANNOTATION

In this article, harmonic functions that extend harmonically on parallel sections are studied. Moreover, are investigated the problems, if the function $u(z,w)$, *harmonic in the domain* $D \times V_r = D \times \{y \in R^m : |y| < r\}$, *and for some subset* $E \subset D$, *for each fixed* x, *the function* $u(x,y)$ *with respect to the variable* y *has singularities only outside a certain thin (discrete or polar) set in* R^m*; and it extends as a harmonic function to the whole space, then of harmonic continuation outside the domain* $D \times V_r$.

KEYWORDS: *Harmonik Functions, Pluripolar Set, Meromorphic Functions, Parallel Sections*

INTRODUCTION

This article explores harmonic functions that extend harmonically on parallel sections. Let $D \subset R^n$ be a domain, and $V_r = \{y : |y| < r\} \subset R^m$.

For simplicity, when $m = 2$, we use the complex plane $C \approx R^2$ instead of R^2.

METHODOLOGICAL APPROACHES

The following problem is considered: let the function (z,w) be defined on $D \times V_r = D \times \{y \in R^m : |y| < r\}$ for each fixed point x on $E \subset D$, the function $u(x,y)$ should be extended harmonically outside the domain, except for certain singularities (discrete or polar) on the whole space R^m .

The question is how to extend the function harmonically outside the domain $D \times V_r$. Specifically, under what conditions on E, the function $u(x,y)$ can be extended harmonically to the whole domain $D \times R^m$, excluding a singular set $S \subset D \times R^m$?

The problem has been studied for holomorphic and pluriharmonic functions by several researchers ([1-5]).
For simplicity, when $m = 2$, we use the complex plane $C \approx R^2$ instead of R^2.

ANALYSIS AND RESULTS

Theorem 1. Let the function $u(x,w)$ be harmonic in

$D \times W_r = D \times \{w = y_1 + iy_2 : |w| < r\} \subset R^n \times C$, and let $C^n = R^n + iR^n$ be the complex space. If

$$E \subset D$$

is a set where for every fixed x^0 , the function $u(x^o, w)$ extends harmonically to the complex plane C outside a specific singular point, then $u(x,w)$ extends harmonically to $(D \times C)$ except for a singular set $S \subset (D \times C)$, here S is a region where

$$\widehat{D} \subset C^n$$

intersects with the $\widehat{S}$ graph of a meromorphic function $w = \varphi(z)$.

In this context, $\widehat{D}$ is a set in C^n, and $D \subset R^n$is an arbitrary harmonic function.
The set $\widehat{D} \supset D$ is a domain where it can be continued holomorphically ([3])

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Proof. According to [3], there exists a region $\widehat{D} \subset C^n$ such that $D \subset \widehat{D}$ and the function $\frac{\partial u(x,w)}{\partial w}$ extends holomorphically to the region $\widehat{D} \times W_r$, where for each fixed $x \in E$, the function $f(x,w) = \frac{\partial u(x,w)}{\partial w}$ extends holomorphically to the entire complex plane C except for one special point.

Now, by A. S. Sadullaev and E. M. Chirka's theorem [2] on the holomorphic continuation of functions along a fixed direction, the function $f(x,w)$ extends holomorphically to $(\widehat{D} \times C)\backslash\hat{S}$ where $\hat{S}: w = \varphi(z)$ is an analytic graph.

Let $\Phi(x,w) = \int_{[0,w]} f(x,w)$ be the initial function for $f(x,w) = \frac{\partial u(x,w)}{\partial w}$ where the integral is taken along any rectifiable contour in $C\backslash\varphi(x)$ for a fixed $x \in D$. In this case, $\Phi(x,w)$ represents a multivalued analytic function.

Now, we use the following fact from [3]: if $\gamma \subset C\backslash\varphi(x)$ is a rectifiable curve connecting two points 0 and w, then

$$\int\limits_{\gamma} \frac{\partial u(x,w)}{\partial w} dw == \int\limits_{\gamma} \frac{\partial u(x,w)}{\partial y_1} dy_1 + \frac{\partial u(x,w)}{\partial y_2} dy_2 + i \int\limits_{\gamma} \frac{\partial u(x,w)}{\partial y_1} dy_2 - \frac{\partial u(x,w)}{\partial y_2} dy_1$$

$$= u(x,w) - u(x,0) + i \int\limits_{\gamma} \frac{\partial u(x,w)}{\partial y_1} dy_2 - \frac{\partial u(x,w)}{\partial y_2} dy_1$$

Thus, we can deduce that $\mathrm{Re}\Phi(x,w) = u(x,w) - u(x,0)$.

Therefore, the function $u(x,w) = \mathrm{Re}\Phi(x,w) + u(x,0)$ represents a multivalued harmonic function in general. In this case, E represents the set of unique solutions for real analytic functions (in particular, for harmonic functions).

Hence, both $\mathrm{Re}\Phi(x,w)$ and $u(x,w)$ are single-valued harmonic functions in the region $(D \times C)\backslash S$, where $S = \hat{S} \cap (D \times C)$.

Indeed, by using the reflection $(x,w) \to \left(x, \frac{w}{\varphi(x)}\right)$ we can view $\boldsymbol{\Pi} = \{x \in D: \varphi(x) = \infty\}$ and the set $= \{(x,w): x \in D: w = 1\}$**.** Hence, there exists a closed rectifiable curve $\gamma \subset C\backslash\{1\}$ such that

$$\mathrm{Re} \oint_{\gamma} \frac{\partial u(x, w\varphi(x))}{\partial w} dw \neq 0 \qquad (1)$$

in $D\backslash\boldsymbol{\Pi}$.
On the other hand, this integral represents a real analytic function in $D\backslash\boldsymbol{\Pi}$, which equals zero for all x in the set $E\backslash\boldsymbol{\Pi}$. Since E is the set of unique solutions for real analytic functions in $D\backslash\boldsymbol{\Pi}$, the fact that the (1) integral is exactly zero expresses the single-valuedness of the function $u(x,w)$. Thus, the proof of the theorem is complete.

In the same manner, the following two theorems may be are proved.

Theorem 2. Let the function $u(x,w)$ be harmonic in the domain
$D \times W_r = D \times \{w = y_1 + iy_2 : |w| < r\} \subset R^n \times C$,
and assume that $C^n = R^n + iR^n$ is a subset of the space, where there exists a set $E \subset D$ that is not pluripolar.

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For every fixed point $x^0 \in E$, the function $u(x^o, w)$, with respect to the variable w, is harmonic and extends to the entire complex plane C, except for a finite set of special points.
Then the function $u(x, w)$ extends harmonically to the domain

$$(D \times C) \setminus S,$$

where S is an analytic subset of $D \times C$, with

$$\hat{S} \subset \hat{D} \times C .$$

Theorem 3. Let the function $u(x, w)$ be harmonic in the domain
$D \times W_r = D \times \{w = y_1 + iy_2 : |w| < r\} \subset R^n \times C$,

and assume that for every fixed point $x^0 \in D$, the function $u(x^o, w)$ with respect to the variable w, is harmonic and extends to the entire complex plane C, except for a polar set of special points.
Then the function $u(x, w)$ extends harmonically to the domain

$(D \times C) \setminus S$, where S is a closed pluripolar subset of $D \times C$, with

$$\hat{S} \subset \hat{D} \times C .$$

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