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EVALUATION OF WEIGHTED CONNECTED VERTEX COVER (W-CVC) ON BUTTERFLY GRAPHS

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ABSTRACT
The Weighted Connected Vertex Cover (W-CVC) problem is a fundamental optimization problem in graph theory, combining the constraints of vertex covering, connectivity, and weight minimization. Given a weighted graph, the objective is to determine a minimum-weight vertex subset that covers all edges while inducing a connected subgraph. This paper focuses on the derivation of W-CVC specifically on Butterfly Graphs, a class of structured interconnection networks widely used in parallel computing. By exploiting the recursive and symmetric properties of butterfly graphs, an explicit derivation of the optimal weighted connected vertex cover is presented. The study provides a systematic approach suitable for exam preparation, highlighting key steps, structural insights, and theoretical justification.

INTRODUCTION
Graph theory plays a crucial role in computer science, discrete mathematics, and network design. Among its classical problems, the Vertex Cover Problem is one of the most widely studied due to its theoretical significance and practical applications in areas such as network security, circuit design, and resource allocation. A natural extension of this problem is the Weighted Vertex Cover, where each vertex is assigned a cost, and the goal is to minimize the total weight of the selected vertices.

An additional constraint leads to the Connected Vertex Cover (CVC) problem, where the chosen vertices must form a connected subgraph. Combining both aspects results in the Weighted Connected Vertex Cover (W-CVC) problem, which is computationally more challenging and belongs to the class of NP-hard problems.

This work focuses on the derivation of W-CVC for Butterfly Graphs, which are highly structured graphs commonly used in communication networks and parallel processing architectures. Due to their layered and recursive structure, butterfly graphs provide an excellent framework for analytical derivation rather than purely algorithmic solutions.

The objective of this study is to present an exam-oriented derivation of the W-CVC on butterfly graphs. The approach emphasizes clarity, step-by-step reasoning, and utilization of graph properties such as symmetry, layering, and connectivity. This makes the content especially useful for students preparing for examinations in graph theory and combinatorial optimization.

Definitions
A Butterfly Graph Bn is a layered graph with:
- n+1 levels (from 0 to n)
- each level contains 2^n vertices
- Total vertices: $(n+1)^2$

Figure 1: Butterfly Graph

```
    v1
   /  \
v2---v3---v4
       \  /
        v5
```

Edges: (v1,v2), (v2,v3), (v3,v1) (v3,v

Each vertex is denoted by:

$$(v, i), v \in \{0,1\}n,\ i = 0,1, \ldots, n$$

Edges:

$$E = \{(v1, v2), (v2, v3), (v3, v1), (v3, v4), (v4, v5), (v5, v3)\}$$

Vertex v3v, is the central shared vertex or vertex v3 acts as the central articulation vertex.
Each vertex $vi \in V$ is associated with a weight $wi > 0$

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Assign Weights
Let weights be, $w1, w2, w3, w4, w5 > 0$

Decision Variables
Define binary variables:
$xi = \{1$ if vertex v_i is included in the cover
0,otherwise

Objective Function
The objective is to minimize the total weight of selected vertices:

$$minZ = i = 1\sum 5wixi$$

Vertex Cover Constraints
Each edge must be covered by at least one endpoint.
For the first triangle:

$$x1 + x2 \geq 1$$
$$x2 + x3 \geq 1$$
$$x1 + x3 \geq 1$$

For the second triangle:

$$x3 + x4 \geq 1$$
$$x4 + x5 \geq 1$$
$$x3 + x5 \geq 1$$

Extension to Hybrid Genetic Algorithms:
Although exact solutions exist for butterfly graphs, hybrid genetic algorithms (HGA) can be used for:

1. Larger generalized networks
2. Approximate solutions

HGA combines:
Global search (GA) with Local refinement.

Algorithmic Implications
In general graphs, solving the W-CVC problem is computationally difficult (NP-hard) due to the need to simultaneously satisfy edge coverage and connectivity constraints across many possible vertex subsets. However, in butterfly graphs, the presence of a central articulation vertex (commonly denoted as v3v_3v3) dramatically simplifies the problem.

The articulation vertex acts as a bridge between the two triangular subgraphs. By selecting this vertex, the connectivity requirement is automatically satisfied, eliminating the need for complex connectivity verification mechanisms such as graph traversal or flow-based constraints.

Once the central vertex is included, the original problem decomposes into two independent and much simpler sub-problems—each corresponding to a triangle. In each triangle, the task reduces to selecting one vertex of minimum weight to ensure edge coverage. This decomposition reduces the exponential search space of possible solutions to a constant number of choices.

As a result, the optimal solution for the butterfly graph can be computed directly using a closed-form expression:

$$Z *= w3 + min(w1, w2) + min(w4, w5)$$

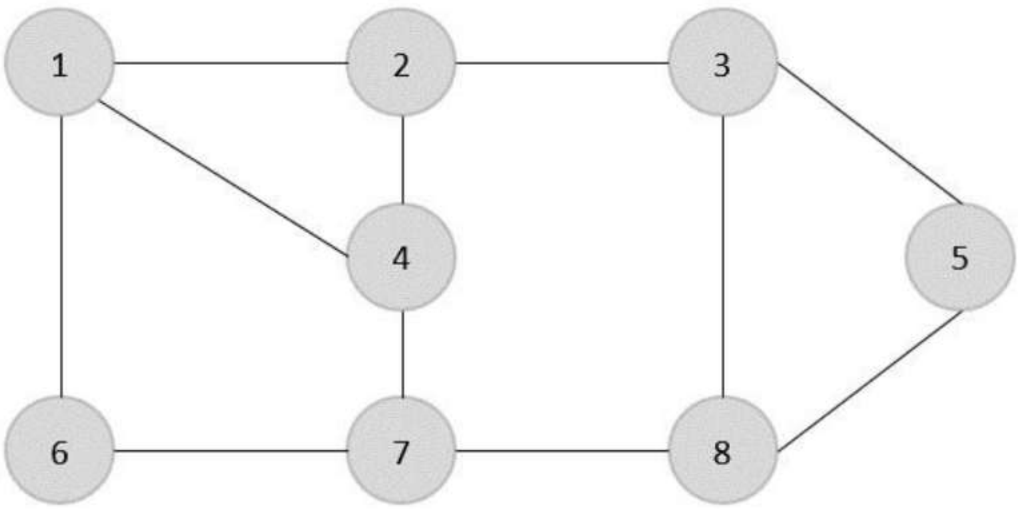
From an algorithmic perspective, this implies that:

- The problem is solvable in constant time O(1)
- No iterative or heuristic optimization methods are required
- The structure enables a deterministic and exact solution

Thus, butterfly graphs illustrate how exploiting graph topology—specifically articulation points and modular decomposition—can transform a generally intractable problem into an efficiently solvable one.
The derived closed-form solution suggests:

Polynomial-time solvability for butterfly graphs with structural decomposition reduces complexity.

Comparison



General W-CVC → NP-hard:
Butterfly graph → efficiently solvable

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Two triangles

- Triangle 1 → $(v1, v2, v3)(v\_1, v\_2, v\_3)(v1, v2, v3)$
- Triangle 2 → $(v3, v4, v5)(v\_3, v\_4, v\_5)(v3, v4, v5)$

```
  v1
 / \
v2---v3---v4
     \ /
      v5
```

v3 is the central articulation vertex

Decision Variables

$xi \in \{0,1\}$

Objective Function

Minimize: $Z = \Sigma\, wi\, xi$

The objective function in W-CVC aims to minimize the total weight of selected vertices while satisfying two essential constraints:

1. Edge Coverage**:** Every edge must be incident to at least one selected vertex.
2. Connectivity: The selected vertices must form a connected subgraph.

Unlike the classical vertex cover problem, where all vertices are treated equally, the weighted version assigns different costs or importance to vertices. Therefore, the objective is not just to minimize the number of vertices but to minimize the overall cost of the solution.

Constraints
In contrast, a butterfly graph possesses a highly structured topology characterized by a central articulation vertex connecting two triangular subgraphs. This structural feature has a profound algorithmic advantage. By simply selecting the articulation vertex, the connectivity requirement is automatically satisfied. The remaining problem decomposes into two independent sub-problems, each involving a small triangle where the optimal choice can be made locally by selecting the minimum-weight vertex.

This structural decomposition eliminates the need for exhaustive search and reduces the computational complexity from exponential to constant time. Unlike general graphs, where sophisticated optimization techniques are required, the butterfly graph admits a direct closed-form solution, making it computationally trivial.

Edge Coverage Constraints

$$x1 + x2 \geq 1\, x2 + x3$$
$$\geq 1\, x1 + x3 \geq 1\, x3 + x4 \geq 1\, x4 + x5 \geq 1\, x3 + x5 \geq 1$$

Connectivity Analysis
The articulation vertex v3 ensures connectivity between the two triangles.

Case 1: x3 = 1 → Connected Case 2: x3 = 0 → Potential disconnection

Thus, enforcing x3 = 1 guarantees connectivity.

Visual Insight from Diagram:
From the diagram:
$v3v\_3v3$ acts as a bridge, removing it disconnects the graph.

Including it simplifies:
Connectivity constraint → satisfied
Coverage → reduced to local choices

Model Reduction
With x3 = 1:
Z = w3 + w1x1 + w2x2 + w4x4 + w5x5

Constraints reduce to:
x1 + x2 ≥ 1 x4 + x5 ≥ 1

The problem decomposes into two independent sub-problems:
$Z* = w3 + min(w1, w2) + min(w4, w5)$

CONCLUSION

This paper presented a complete derivation of the Weighted Connected Vertex Cover problem on butterfly graphs. By exploiting graph structure, the problem simplifies significantly, yielding a closed-form optimal solution. This highlights the importance of structural insights in solving complex combinatorial optimization problems.

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